

[**NUMERICAL RESIDUAL PERTURBATION SOLUTIONS  
APPLIED TO THE PROBLEM OF A CLOSE SATELLITE  
OF THE SMALLER BODY  
IN THE RESTRICTED THREE-BODY PROBLEM**] [

SEPTEMBER 1965

By  
J.C. Walker  
M.C. Eckstein  
H.A. Holman

Prepared under Contract No. NASw-901  
by Douglas Aircraft Company, Inc.  
Missile and Space Systems Division  
Santa Monica, California  
for  
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

## ABSTRACT

12184

The purpose of this report is to demonstrate the new method of numerical residual perturbation solution as applied to the problem of a close satellite of the smaller body in the restricted three-body problem. Cowell demonstrated a method of numerically solving the total differential equations of motion of an orbiting object. The variation of parameters and Encke's methods take advantage of the known analytic solution to the two-body problem and numerically handle only the perturbations to the orbit. This report demonstrates the use of an analytic series perturbation solution as a reference orbit (rather than using conics as a reference) and the numerical solution of the residual (i.e., not accounted for by the analytic solution) perturbation equations of motion. The new method shows advantages over the full region of validity of the analytic reference orbit. The analytic theory of motion used is such as to limit the application of the results to orbits of small inclination and eccentricity with special initial conditions. The program is thus just for demonstration. This work was supported by contract NASw-901.

*Author*

## CONTENTS

|  | <u>Page</u> |
|--|-------------|
| <b>Section 1    INTRODUCTION . . . . .</b>   | 1           |
| <b>Section 2    DEFINITION OF SYMBOLS . . . . .</b>  | 7           |
| <b>Section 3    DEVELOPMENT OF EQUATIONS . . . . .</b>   | 41          |
| 3.0    General . . . . .   | 41          |
| 3.1    Equations of Motion . . . . .   | 41          |
| 3.2    The Jacobi Constant . . . . .   | 43          |
| 3.3    Solutions . . . . .   | 45          |
| 3.4    Method of Numerical Solution of Differential<br>Equations . . . . .   | 78          |
| <b>Section 4    EQUATIONS IN ORDER OF SOLUTION . . . . .</b>   | 85          |
| 4.1    Print Explanation of Error Code . . . . .   | 85          |
| 4.2    Program Input . . . . .   | 85          |
| 4.3    Move Input to Working Storage and Check Input . . . . .   | 86          |
| 4.4    Compute Constants . . . . .   | 87          |
| 4.5    Print Constants . . . . .   | 92          |
| 4.6    Print Header for Numerical Residual Perturbation<br>Solution Listing the Input Values of $a$ , $e$ , $i$ ,<br>and $\mu$ . . . . . | 93          |
| 4.7    Set Perturbation-Total Flag . . . . .   | 93          |
| 4.8    Compute Initial Values of Position, Velocity,<br>and Acceleration . . . . .   | 93          |
| 4.9    Set Run Start Flags . . . . .   | 97          |
| 4.10    Output of Numerical Residual Perturbation<br>Solution . . . . .  | 97          |
| 4.11    Test Halt Flag, KHALT . . . . .  | 98          |
| 4.12    Call Numerical Solution Subroutine for Residual<br>Perturbation Equations . . . . .  | 98          |
| 4.13    Test Runge-Kutta Flag, KR . . . . .  | 99          |
| 4.14    Test Halt Flag, KHALT . . . . .  | 99          |
| 4.15    Compute Two-Body Equations . . . . .   | 99          |
| 4.16    Acceleration Components for Numerical Residual<br>Perturbation Equations of Motion . . . . .                                     | 102         |
| 4.17    Computations for Print Only . . . . .  | 104         |
| 4.18    Save Plot Values and Test if Plotting Array<br>is Full . . . . .   | 116         |
| 4.19    Initiate Numerical Solution of Total Equations<br>of Motion . . . . .  | 117         |
| 4.20    Print Numerical Solution of Total Equations<br>of Motion . . . . .   | 119         |
| 4.21    Test Halt Flag, KHALT . . . . .  | 120         |

|  | <u>Page</u> |
|--|-------------|
| 4.22 Call Numerical Solution Subroutine for Total<br>Equations of Motion . . . . .             | 120         |
| 4.23 Test Runge-Kutta Flag, KR . . . . .   | 121         |
| 4.24 Test Halt Flag, KHALT . . . . .   | 121         |
| 4.25 Compute Total Equations of Motion . . . . .   | 121         |
| 4.26 Test Runge-Kutta Flag, KR . . . . .   | 122         |
| 4.27 Test Print Flag, IPRINT . . . . .   | 122         |
| 4.28 Compute Data for Print and for Plot Routine . . . . .                                     | 122         |
| 4.29 Test Halt Flag, KHALT . . . . .   | 125         |
| 4.30 Test if Printing is Desired . . . . .   | 125         |
| 4.31 Call the Plot Subroutine . . . . .  | 125         |
| 4.32 Run is Completed, Call Next Case at Step 4.2 . . . . .                                    | 126         |
| <br>Section 5 MAIN PROGRAM DETAIL FLOW CHARTS . . . . .  | 127         |
| 5.1 Explanation . . . . .  | 127         |
| <br>Section 6 MAIN PROGRAM LISTING . . . . .   | 135         |
| <br>Section 7 SUBROUTINE FOR NUMERICAL SOLUTION OF DIFFERENTIAL<br>EQUATIONS, RKSIMP . . . . . | 149         |
| 7.1 Equations in Order of Solution . . . . .   | 149         |
| 7.2 Subroutine RKSIMP Detail Flow Charts . . . . .   | 167         |
| 7.3 RKSIMP Listing . . . . .   | 171         |
| <br>Section 8 SUBROUTINE PLOT . . . . .  | 179         |
| 8.1 Development of Equations . . . . .   | 179         |
| 8.2 Equations in Order of Solution . . . . .   | 180         |
| 8.3 Subroutine PLOT Detail Flow Charts . . . . .   | 191         |
| 8.4 PLOT Listing . . . . .   | 195         |
| <br>Section 9 INPUT AND OUTPUT . . . . .   | 199         |
| 9.1 Input . . . . .  | 199         |
| 9.2 Output . . . . .   | 201         |
| <br>Section 10 DISCUSSION OF RESULTS . . . . .   | 209         |
| REFERENCES . . . . .   | 214         |
| APPENDIX . . . . .   | 215         |

## FIGURES

|  | <u>Page</u> |
|--|-------------|
| 1      Overall Program Logic . . . . .   | 5           |
| 2      FORTRAN Data Load Sheet . . . . . | 200         |

TABLE

Section 1  
INTRODUCTION

The purpose of this paper is to show the relative merits for prediction of future position of a satellite of the smaller body in the restricted three-body problem, of the analytic solution of reference 1, of numerical solution of the residual perturbation differential equations not accounted for by the theory of motion in reference 1, of the perturbation differential equations from a Keplerian ellipse, and of the total equations of motion.

In a numerical solution, the time interval is normally chosen so as to make the error due to series truncation in the approximating function approximately equal to the error due to truncation of numbers in the computation of the equations of motion and the approximating function. The balancing of these errors assures maximum accuracy from the method. (Actually, to determine mathematically the time interval, it was necessary in this program to make the series truncation error slightly larger than number truncation error.) In a numerical solution of the total equations, if  $\delta Y$  is added to  $Y_i$  at each point, then the error builds up as:

$$\begin{array}{r} Y_i = \text{xxxxxxxx} \\ + \delta Y = \text{xxxxxxxx} \\ \hline = Y_{i+1} = \text{xxxxxxxx} \end{array}$$

digit affected by series truncation  
digit affected by number truncation  
digit where error accumulates

If it is possible to write  $Y$  as  $Y_a + \Delta Y$  where  $Y_a$  is obtained by an analytic approximation and only  $\Delta Y$  is obtained by numerical solution of the differential equations, then the error accumulates as:

$$\begin{aligned}
 \Delta Y_i &= \text{xxxxxxxx} \\
 + \delta Y_i &= \text{xxxxxxxx} \\
 \hline
 = \Delta Y_{i+1} &= \text{xxxxxxxx} \\
 + Y_{ai+1} &= \text{xxxxxxxx} \\
 \hline
 = Y_{i+1} &= \text{xxxxxxxxxx} \\
 &\quad \downarrow \quad \text{digit where error accumulates} \\
 &\quad \downarrow \quad \text{digit where number truncation occurs} \\
 &\quad \quad \quad \text{but is not accumulated.}
 \end{aligned}$$

It is, of course, obvious that the better the approximation  $Y_a$ , the smaller  $\Delta Y$  will be and the slower the accumulation of number truncation.

This approach in this paper differs from the work of Encke primarily in that an approximate analytic solution of the perturbed motion is used as  $Y_a$  rather than  $Y_a$  being an unperturbed Kepler ellipse. The procedure assumes a differential equation of the form:

$$\ddot{Y} = f(X, Y, \dot{Y}) + g(X, Y, \dot{Y}) \quad (\text{I-1})$$

It is approximated by:

$$\ddot{Y} = f(X, Y, \dot{Y}) \quad (\text{I-2})$$

and equation (I-2) is solved analytically to yield the approximation  $Y_a$  in the form of a finite asymptotic series. The exact solution,  $Y$ , of equation (I-1) is written in the form:

$$Y = Y_a + Z \quad (\text{I-3})$$

Equation (I-3) is differentiated twice and the result substituted for the left-hand side of equation (I-1). The value of  $Z$  is computed by a numerical

solution of this differential equation. The solution,  $Y$ , involves two approximations in this case. First, it is derived from equation (I-2), not equation (I-1); second, the truncation of the series for  $Y_a$  represents an error. It may be seen that the numerical procedure for  $Z$  corrects for both approximations.

The above discussion is independent of whether the machine computations are done single-precision, double-precision, or n-precision since numerical solution of the total equations of motion, of the perturbations from a conic, and of the residual perturbation from a theory may all be done to n-precision without changing the relative advantages.

Reference 1 was an early work in the series of papers on celestial mechanics described in the list of references. Besides being limited to small eccentricities and inclinations, it does not generalize initial conditions, but always begins with the perigee and node lined up along the Earth-Moon line. Therefore, rectification such as Encke used is not possible. In using it as a basis for the numerical solution, there results a demonstration for the new numerical techniques but not a useful analytic tool. For demonstration purposes, it was not deemed worth-while to attempt the cancellation of all oscillatory terms of the highest order carried by the approximation in order to provide the minimum residual perturbations. What has been done in this paper is to use the precessing elliptic orbit which provides the zero order portion of reference 1 as the reference orbit. This is an appreciable improvement over a fixed initial osculating ellipse and serves to demonstrate the usefulness of the technique.

In this example, the equations of motion are those of the restricted three-body problem. No other perturbations are considered. However, no real increase in complexity or decrease in accuracy result from adding other perturbations. This is being demonstrated in a companion report for a satellite of an oblate planet. In the companion report, an analytic approximation is used for  $J_2$  and  $J_4$  perturbations;  $J_3$  and higher-order Earth potential terms as well as Sun and Moon perturbations are treated numerically.

This report consists of a description of the program, a discussion of the results, and plotted results for a variety of mass ratios, semi-major axes, eccentricities, and inclinations. The general flow of the program is illustrated in figure 1. The subsection numbers on the top of each block refer to Section 4 where the equations are detailed.

In this report, the equal sign is often used in the FORTRAN sense, i.e.,

$$x = x + 1$$

means replace  $x$  by  $x + 1$ . Similarly, symbols such as  $A(N)$  are used to refer to an array of  $N$  numbers.

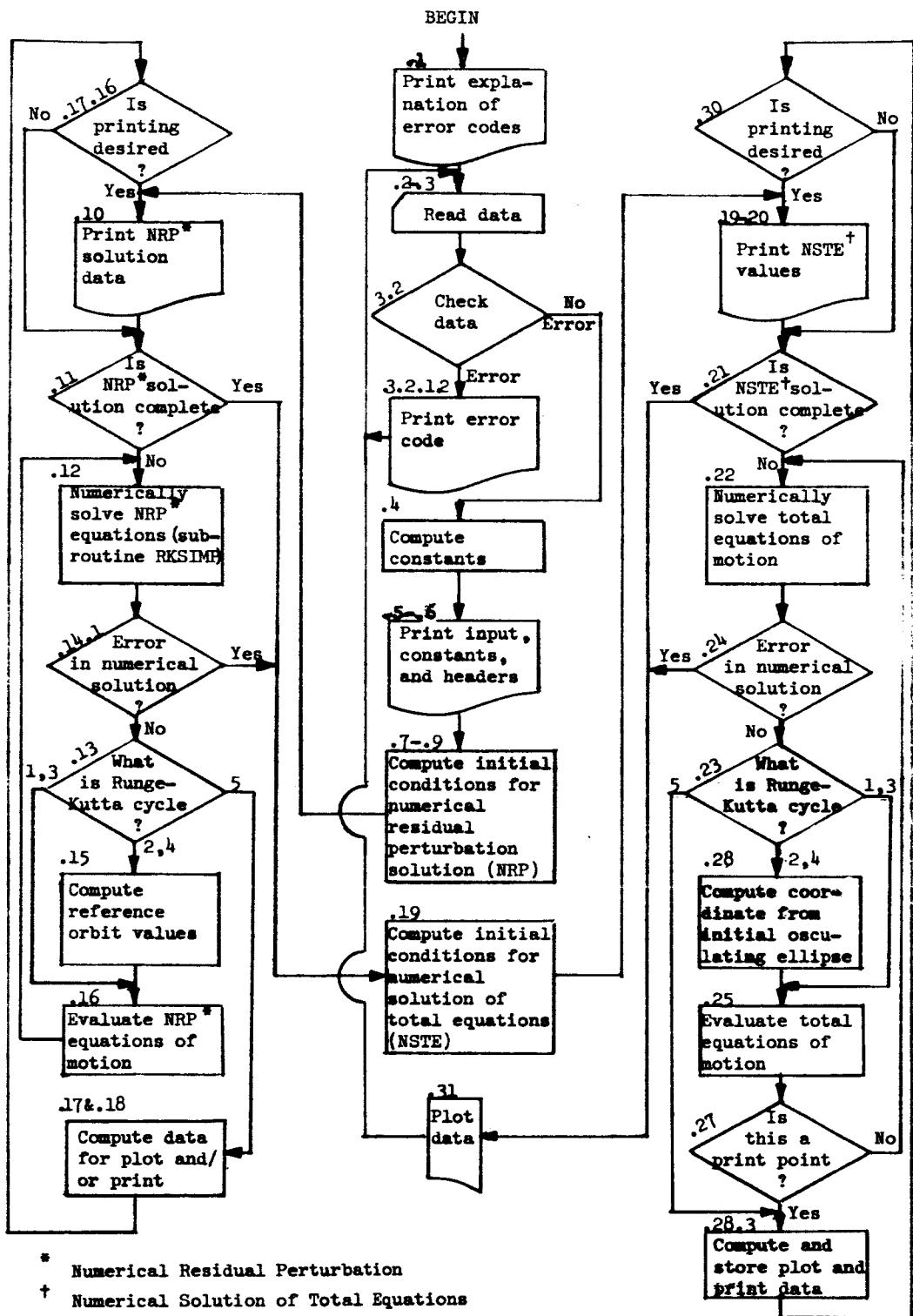


Figure 1

Section 2  
DEFINITION OF SYMBOLS

|               |  |
|---------------|--|
| $a$           | Semi-major axis                                  |
| $a$           | Subscript for analytic solution from reference 1 |
| $a_1$         | Semi-major axis in analytic solution = A1        |
| $a_2$         | $a^2$  |
| $a_3$         | $a^3 = A3$                                       |
| $a_{12}$      | $\sqrt{a}$                                       |
| $a_{32}$      | $a^{3/2}$  |
| $a_{92}$      | $a^{9/2} = A92$                                  |
| $a_e$         | $a(1-e^2) = AEC$                                 |
| $a_{e1/2}$    | $\sqrt{a(1-e^2)}$                                |
| $A_p$         | Plotting array of dimension 13,000 with elements |
| $ap_{1+13i}$  | $(\Delta C/C)_{ai}$                              |
| $ap_{7+13i}$  | $(\Delta \rho/a)_{pki}$                          |
| $ap_{2+13i}$  | $(\Delta C/C)_{pki}$                             |
| $ap_{8+13i}$  | $(\Delta \rho/a)_{ci}$                           |
| $ap_{3+13i}$  | $(\Delta C/C)_{ci}$                              |
| $ap_{9+13i}$  | $(\Delta \rho/a)_{ki}$                           |
| $ap_{4+13i}$  | $(\Delta C/C)_{ki}$                              |
| $ap_{10+13i}$ | $(\Delta \rho a^{1/2})_{ai}$                     |
| $ap_{5+13i}$  | $(\Delta C/C)_{pi}$                              |
| $ap_{11+13i}$ | $(\Delta \rho a^{1/2})_{pki}$                    |
| $ap_{6+13i}$  | $(\Delta \rho/a)_{ai}$                           |
| $ap_{12+13i}$ | $(\Delta \rho a^{1/2})_{ci}$                     |
| $ap_{13+13i}$ | $(\Delta \rho a^{1/2})_{ki}$                     |
| $a_s$         | Semi-major axis of precessing ellipse            |

$i = 0, 1, \dots, 999$

|           |  |
|-----------|--|
| $a_{s32}$ | FORTRAN symbol for $a_s^{3/2}$   |
| A1        | FORTRAN symbol for semi-major axis, $a$  |
| A2        | FORTRAN symbol for $a^2$   |
| A3        | FORTRAN symbol for $a_3 = a^3$   |
| A12       | FORTRAN symbol for $\sqrt{a}$  |
| A32       | FORTRAN symbol for $a^{3/2} = a_{32}$  |
| A92       | FORTRAN symbol for $a^{9/2} = a_{92}$  |
| ACOBI     | FORTRAN symbol for the Jacobi integral - $1/\epsilon^2 r$<br>(Computed at initial time using numerical residual<br>perturbation solution values) |
| AEC       | FORTRAN symbol for $a(1-e^2)$  |
| AESQ      | FORTRAN symbol for $\sqrt{a_s(1-e_s^2)}$   |
| ALPHAL    | FORTRAN symbol for $\alpha_1$  |
| ALPHA2    | FORTRAN symbol for $\alpha_2$  |
| AP        | FORTRAN symbol for $A_p$   |
| AP(I)     | FORTRAN symbol for $a_{pi}$  |
| ARNODE    | FORTRAN symbol for argument of node, $\Omega$  |
| AS        | FORTRAN symbol for $a_s$   |
| AS32      | FORTRAN symbol for $a_s^{3/2}$   |

|          |   |
|----------|---|
| B1       | FORTRAN symbol for $c_{15} + \rho_s c_{16}$   |
| B2       | FORTRAN symbol for $c_{17} + \rho_s c_{18}$   |
| BETA     | FORTRAN symbol for $\beta$  |
| C        | FORTRAN symbol for the Jacobi constant  |
| c        | Subscript for values obtained from numerical solution<br>of total equations of motion           |
| $c_1$    | $\frac{15}{4} \epsilon a^3 e$   |
| $c_2$    | $\frac{15}{4} \epsilon a^{1/2} e$   |
| $c_3$    | $\frac{\frac{3}{8} \epsilon i a^{5/2}}{1 - \epsilon a^{3/2}}$                                   |
| $c_6$    | $\dot{\Omega}$  |
| $c_7$    | $\epsilon^4 \lambda_1^2 = \dot{\Omega}^2$   |
| $c_8$    | $\frac{4}{1 - \epsilon}$  |
| $c_9$    | $a^{3/2} (1 - e^2)^{3/2}$   |
| $c_{10}$ | $\frac{\epsilon^2 \omega_1}{1 + \epsilon^2 \omega_1} + \epsilon_1 a^{3/2} (1 + \epsilon_1 m_1)$ |
| $c_{11}$ | $2P_1 - 1$  |
| $c_{12}$ | $2\dot{\Omega} / \sqrt{a(1 - e^2)} = 2\dot{\Omega}/a_{el/2}$                                    |
| $c_{13}$ | $2\dot{\omega} / \sqrt{a(1 - e^2)} = 2\dot{\omega}/a_{el/2}$                                    |

$$c_{14} \quad 2C_{\omega\dot{\Omega}} = 2\dot{\omega}\ddot{\Omega}$$

$$c_{15} \quad - \frac{2}{ae^{1/2}} (\dot{\Omega} + \dot{\omega} \cos i)$$

$$c_{16} \quad - 2\dot{\omega}\ddot{\Omega} - (\dot{\omega}^2 + \dot{\Omega}^2) \cos i$$

$$c_{17} \quad \frac{2}{ae^{1/2}} (\dot{\Omega} \cos i + \dot{\omega})$$

$$c_{18} \quad 2\dot{\omega}\ddot{\Omega} \cos i + \dot{\omega}^2 + \dot{\Omega}^2$$

$$c_{19} \quad \dot{\omega}^2 \sin i$$

$$c_{20} \quad \frac{2e \sin i}{ae^{1/2}} \dot{\omega}$$

$$c_{21} \quad \frac{2 \sin i}{ae^{1/2}} \dot{\omega}$$

$$c_{22} \quad - \frac{2e}{ae^{1/2}} (\dot{\Omega} + \dot{\omega} \cos i)$$

$$c_{23} \quad \frac{2e}{ae^{1/2}} (\dot{\Omega} \cos i + \dot{\omega})$$

$$c_{24} \quad \sqrt{1-e_s^2}$$

$$c_E \quad \cos E$$

$$c_i \quad \cos i$$

$$c_{\text{init}} \quad \text{Initial value of Jacobi constant}$$

|                        |  |
|------------------------|--|
| $C_{\max}$             | Maximum of absolute values of the components of the velocity increment over the last two Runge-Kutta cycles = CMAX |
| $C_{P1\psi}$           | $\cos (2p_1 - 1)\psi$  |
| $C_{\text{part}}$      | Part of Jacobi constant (see Section 3, equation (68))   |
| $C_v$                  | $\cos v$   |
| $C_{\epsilon t}$       | $\cos \epsilon t$  |
| $C_{\theta_1}$         | $\cos \theta_1$  |
| $C_{\phi_s}$           | $\cos \phi_s = \cos (v + \omega)$  |
| $C_\psi$               | $\cos \psi$  |
| $C_\Omega$             | $\cos \Omega$  |
| $C_\omega$             | $\cos \omega$  |
| $C_{\omega\Omega}$     | $\dot{\omega}\dot{\Omega}$   |
| $C_{\omega^2\Omega^2}$ | $\dot{\omega}^2 + \dot{\Omega}^2$  |
| $C_1$ through $C_3$    | FORTRAN symbol for $C_1$ through $C_3$   |
| $C_6$ through $C_{21}$ | FORTRAN symbol for $C_6$ through $C_{21}$  |
| CALPHA                 | FORTRAN symbol for $\cos \alpha_1$   |
| CC11PS                 | FORTRAN symbol for $\cos (C_{11}\psi)$   |

|                 |  |
|-----------------|--|
| CEC             | FORTRAN symbol for cos E   |
| CEPT            | FORTRAN symbol for cos et  |
| CI              | FORTRAN symbol for cosine inclination, cos i   |
| CIS             | FORTRAN symbol for cos i <sub>s</sub>  |
| CMAX            | FORTRAN symbol for the maximum of absolute values<br>of the components of the velocity increment over the<br>last two Runge-Kutta cycles = C |
| CNO             | FORTRAN symbol for cos Ω   |
| COPER           | FORTRAN symbol for cos ω <sub>s</sub>  |
| COV             | FORTRAN symbol for cos v <sub>s</sub>  |
| CPI             | FORTRAN symbol for cos (2p <sub>1</sub> - 1)ψ  |
| CP <sub>3</sub> | FORTRAN symbol for cos p <sub>3</sub> ψ  |
| CPIPS           | FORTRAN symbol for cos (p <sub>1</sub> ψ + α <sub>1</sub> )  |
| CPIPST          | FORTRAN symbol for cos (p <sub>1</sub> ψ)  |
| CPS             | FORTRAN symbol for cos ψ   |
| CSNO            | FORTRAN symbol for cos Ω <sub>s</sub>  |
| CTHETA          | FORTRAN symbol for cos θ <sub>1</sub>  |
| CTRL            | FORTRAN symbol for cos (v + ω <sub>s</sub> )   |

|                        |   |
|------------------------|---|
| $D_n$                  | $1 + e \cos E$  |
| DEGI                   | FORTRAN symbol for $i$ (degrees)  |
| DELM                   | FORTRAN symbol for the error in mean anomaly when solving Kepler's equation   |
| DELXD, DELYD,<br>DELZD | In the Runge-Kutta subroutine, FORTRAN symbols for the increment of the components of velocity accumulated for the time interval computation  |
| DEN                    | FORTRAN symbol for $D_n$  |
| DNOD                   | FORTRAN symbol for $\dot{\Omega} = C_6$   |
| DPDN                   | FORTRAN symbol for $\ddot{\omega\Omega} = C_{\omega\Omega}$   |
| DPER                   | FORTRAN symbol for $\dot{\omega}$   |
| D2PHDT                 | FORTRAN symbol for $d^2\phi/dt^2$   |
| DPHIDT                 | FORTRAN symbol for $d\phi/dt$   |
| DPNSQ                  | FORTRAN symbol for $\dot{\omega}^2 + \dot{\Omega}^2 = C_{\omega\Omega}^2$   |
| D2PSDT                 | FORTRAN symbol for $d^2\psi/dt^2$   |
| DPSIDT                 | FORTRAN symbol for $d\psi/dt$   |
| DR                     | FORTRAN symbol for $(\Delta x^2 + \Delta y^2 + \Delta z)^2$   |
| DRC                    | FORTRAN symbol for the magnitude of the vector position difference between the numerical residual perturbation solution and the fixed initial osculating ellipse solution at a given time |

|        |  |
|--------|--|
| DRDC   | FORTRAN symbol for the magnitude of the vector velocity difference between the numerical residual perturbation solution and the numerical solution of the total equation's of motion at a given time |
| DRDF   | FORTRAN symbol for the magnitude of the vector velocity difference between the numerical residual perturbation solution and the initial osculating ellipse value at a given time                     |
| DRF    | FORTRAN symbol for the magnitude of the vector position difference between the numerical residual perturbation solution and the initial osculating ellipse value at a given time                     |
| DSDPSI | FORTRAN symbol for $ds/d\psi$  |
| DSDT   | FORTRAN symbol for $ds/dt$   |
| DT     | FORTRAN symbol for $\Delta t$  |
| DT2    | FORTRAN symbol for $\Delta t/2$  |
| DT3    | FORTRAN symbol for $\Delta t/3$  |
| DT4    | FORTRAN symbol for $\Delta t/4$  |
| DT6    | FORTRAN symbol for $\Delta t/6$  |
| DT1DPS | FORTRAN symbol for $\epsilon^2 \left( \frac{dt}{d\psi} \right)_1$  |
| DTDPSI | FORTRAN symbol for $dt/d\psi$  |
| DTM    | FORTRAN symbol for multiplier on $\Delta t$ if error is less than the minimum allowable = 1.5 if input $W_9 \leq 1$  |
| DTODPS | FORTRAN symbol for $\left( \frac{dt}{d\psi} \right)_0$   |

|                  |   |
|------------------|---|
| DX, DY, DZ,      | FORTRAN symbol for $\Delta x, \Delta y, \Delta z$   |
| DXD, DYD, DZD,   | $\dot{\Delta x}, \dot{\Delta y}, \dot{\Delta z}$  |
| DX2D, DY2D, DZ2D | $\ddot{\Delta x}, \ddot{\Delta y}, \ddot{\Delta z}$   |
|                  | position, velocity and acceleration obtained from residual perturbation equations. Coordinate system references to plane of planetary motion.   |
| E                | Eccentric anomaly in main program (measured from apogee). In subroutine RKSIMP, the maximum component of the difference between the Runge-Kutta and Simpson rule solutions for velocity |
| e                | Eccentricity (initial)  |
| $e_2$            | $\sqrt{1-e^2}$  |
| $E_{\min}$       | Minimum error in Runge-Kutta routine  |
| $E_n$            | Tenative guess at eccentric anomaly   |
| $E_{r_1}$        | $(1-\epsilon^4)\epsilon^2/r^3$  |
| $E_{r_2}$        | $(1-\epsilon^4)\epsilon^2[2(x' \cos \epsilon t + y' \sin \epsilon t) + \epsilon^2 \rho^2] \cdot \frac{[1+r(1+r)]}{r^3(1+r)}$  |
| EN               | FORTRAN symbol for $E_n$  |
| $e_s$            | Eccentricity of precessing ellipse  |
| $E_{2x}$         | $\epsilon_x^2$  |
| $e_2$            | $\sqrt{1-e^2} = EC2$ ; (also temporary value of $1-e^2$ )   |
| $e_3$            | $1 + e$   |

|       |   |
|-------|---|
| EALL  | FORTRAN symbol for maximum allowable error in Runge-Kutta = $ w_8 $<br>set = $10^{-7}$ if input $w_8 = 0$                                   |
| EC1   | FORTRAN symbol for e  |
| EC2   | FORTRAN symbol for $e_2 = \sqrt{1-e^2}$ ; (also temporary value of $1-e^2$ in Sections 4.3.22 - 4.4)  |
| EC3   | FORTRAN symbol for $1+e$  |
| ECAN  | FORTRAN symbol for eccentric anomaly (measured from apogee) in analytic solution  |
| ELMB  | FORTRAN symbol for $\lambda_1$  |
| EM    | FORTRAN symbol; in subroutine PLOT, the time at the beginning of each series of $\Delta C/C$ , $\Delta p/a$ , $\Delta \dot{p}a^{1/2}$ plots |
| EN    | FORTRAN symbol; in subroutine PLOT, the time at the end of each series of $\Delta C/C$ , $\Delta p/a$ , $\Delta \dot{p}a^{1/2}$ plots       |
| EP1   | FORTRAN symbol for $\epsilon$   |
| EP2   | FORTRAN symbol for $\epsilon^2$   |
| EP4   | FORTRAN symbol for $\epsilon^4$   |
| EPA   | FORTRAN symbol for $\epsilon a^{3/2}$   |
| EPR1  | FORTRAN symbol for $(1-\epsilon^4)\epsilon^2/r^3$   |
| EPR2  | FORTRAN symbol for $(1-\epsilon^4)(1 - 1/r^3)$  |
| EP2X  | FORTRAN symbol for $\epsilon^2 x = E_{2X}$  |
| ERMIN | FORTRAN symbol for the minimum allowable error = $ w_{12} $   |

|         |  |
|---------|--|
| ERROR   | FORTRAN symbol for $t(\psi) - t$ (should equal zero)   |
| ES      | FORTRAN symbol for $e_s$   |
| ESA     | FORTRAN symbol for the eccentric anomaly of two-body solution  |
| ESTER   | FORTRAN symbol for estimated error in Runge-Kutta subroutine   |
| f       | $\frac{1 - (1 + 2q)^{-3/2}}{q} = \text{Encke's } f \text{ series}$   |
| FAC(40) | FORTRAN symbol for the factor in polynomial expression for $f$   |
| FTD     | FORTRAN symbol for a factor ( $< 1$ ) by which the calculated computing interval is multiplied before use to protect against round-off error   |
| FPS(I)  | FORTRAN symbol for $F_{\psi i}$  |
| FQ      | FORTRAN symbol for $f_q = 1 - (1 + 2q)^{-3/2}$ See Section 3, equation (102)   |
| FRA(40) | FORTRAN symbol for the fraction by which a term in series for $f$ must be multiplied to give succeeding term   |
| G       | $-\frac{\frac{dt_o}{d\psi} \epsilon^2 \omega_1 + \epsilon^2 \frac{dt_1}{d\psi} (1 + \epsilon^2 \omega_1)}{\frac{dt}{d\psi} \frac{dt_o}{d\psi} (1 + \epsilon^2 \omega_1)} - \epsilon^2 \omega_1 - \dot{\omega}$ |
| GAMMA   | FORTRAN symbol for $\epsilon t$ modulo $2\pi$  |

|                  |   |
|------------------|---|
| GNU1             | FORTRAN symbol for $v_1$  |
| HX, HY, HZ       | FORTRAN symbols for dummy variables of position,<br>velocity, and acceleration in Runge-Kutta subroutine.   |
| HXD, HYD, HZD    | Equivalent to:  |
| HX2D, HY2D, HZ2D | $\Delta\vec{x}$ , $\dot{\Delta}\vec{x}$ , $\ddot{\Delta}\vec{x}$ for numerical residual perturbation<br>solution  |
|                  | $\vec{x}$ , $\dot{\vec{x}}$ , $\ddot{\vec{x}}$ for numerical solution of total equations<br>of motion   |
| HXDA, HYDA, HZDA | FORTRAN symbol for Simpson's rule; integrated values<br>of $\vec{x}$ , $\vec{y}$ , $\vec{z}$ , respectively   |
| I                | FORTRAN symbol, main program cards 220-230 -- index<br>for zeroing input array W  |
|                  | Main program cards 563-564 -- index for zeroing first<br>13 elements of plotting array AP   |
|                  | Main program card 1030 -- index for printing input<br>array W   |
|                  | Main program cards 2510-2660 -- index for Encke's f<br>series   |
|                  | Main program card 5380 -- index used to print plot-<br>table data at computing point ITD  |
|                  | Runge-Kutta subroutine cards 6670-6680 -- index for<br>restoring initial condition array S from array SP<br>after print point, which is not a normal computing<br>point while integrating total equations of motion |
|                  | Runge-Kutta subroutine cards 7140-7150, 7670-7680,<br>and 7850-7860 -- index for restoring initial condi-<br>tion array S from array SS when computing interval<br>fails  |

|           |  |
|-----------|--|
| i         | Inclination (initial) held constant in analytic solution (radians)   |
| $i^\circ$ | Inclination (degrees)  |
| $i_s$     | Inclination of precessing ellipse  |
| ICH       | FORTRAN symbol for the flag to signal which equations are being numerically solved   |
|           | Residual perturbation equations if ICH = 1   |
|           | Total equations of motion if ICH = 2   |
| IERR      | FORTRAN symbol; in subroutine PLOT for number of off-scale points  |
| II        | FORTRAN symbol; in subroutine PLOT, an index giving the plot point number of the first point of each series of $\Delta C/C$ , $\Delta \rho/a$ , $\Delta \dot{\rho}a^{1/2}$ plots |
| III       | FORTRAN symbol; in subroutine PLOT, an index giving the plot point number of the last point of each series of $\Delta C/C$ , $\Delta \rho/a$ , $\Delta \dot{\rho}a^{1/2}$ plots  |
| IIK       | FORTRAN symbol; in subroutine PLOT, an index for successive sets of $\Delta C/C$ , $\Delta \rho/a$ , $\Delta \dot{\rho}a^{1/2}$ plots  |
| IP        | FORTRAN symbol for the initial point flag  |
|           | IP = 1 on first point  |
|           | IP = 2 on all others   |
|           | Set = 1 by main program  |
|           | Set = 2 by Runge-Kutta subroutine  |

IPRINT FORTRAN symbol for print flag  
 IPRINT = 1 to call for print  
 IPRINT = 2 to suppress print  
 IPRINT = 3 in integration of total equations of motion, designates a print point that is also a regular computed point

IS FORTRAN symbol for  $i_s$  (a floating point number)

IT FORTRAN symbol for the index for first element in plot array AP for computing point ITP  
 $IT = 13ITP - 12$

ITP FORTRAN symbol for the index for plotting array

ITQ FORTRAN symbol for the index for last element in plot array AP for computing point ITP  
 $ITQ = IT + 12$

J FORTRAN symbol for main program cards 2570-2590 -- index used to compute coefficients in Encke's f series

JACOBI FORTRAN symbol for the initial value of Jacobi constant (a floating point variable)

K FORTRAN symbol for the index used to compute coefficients of Encke's f series = I - J

K FORTRAN symbol; in subroutine PLOT, index to locate proper values in AP array for each plot. When

$K = -12 + 13I$   $\Delta C/C$  values are plotted  
 $K = -7 + 13I$   $\Delta \rho/a$  values are plotted  
 $K = -3 + 13I$   $\Delta \rho a^{1/2}$  values are plotted

|       |  |
|-------|--|
| K     | In subroutine PLOT, subscript for initial osculating ellipse values  |
| KC    | FORTRAN symbol for flag to signal Simpson's rule computations<br><br>No if KC = 1<br>Yes if KC = 2   |
|       | Under control of Runge-Kutta subroutine  |
| KF    | FORTRAN symbol for intermediate failure counter in Runge-Kutta subroutine  |
| KFAIL | FORTRAN symbol for failure counter in Runge-Kutta subroutine   |
| KHALT | FORTRAN symbol for halt flag<br><br>If = 1, run continued<br>If = 2, run halted, next case called<br>If = 3, perturbed trajectory completed, numerical solution of total equations of motion to be initiated |
| KR    | FORTRAN symbol for Runge-Kutta flag to signal which pass of Runge-Kutta. Initially set = 1 in main program; thereafter under control of Runge-Kutta subroutine   |
| KT    | FORTRAN symbol for computing interval counter  |
| L     | FORTRAN symbol for index used in computing coefficients of Encke's f series = I - 1  |

L

FORTRAN symbol; in subroutine PLOT, an index used to establish for which plot the grid and common labeling are being done

$$L = 1 \Delta C/C$$

$$L = 2 \Delta \rho/a$$

$$L = 3 \Delta \dot{\rho}a^{1/2}$$

M

Mean anomoly

 $m_1$ 

$$\frac{-a_{3/2}(1 + 12e)}{6}$$

M1

FORTRAN symbol for  $m_1$

MFAIL

FORTRAN symbol for the maximum number of failures allowed in Runge-Kutta subroutine =  $|W_{11}|$

MRKPT

FORTRAN symbol; in subroutine PLOT, an array to identify the plotting symbols to be used for subroutine APLOTV

| <u>Term No.</u> | <u>Value</u> | <u>Symbol Indicated</u> | <u>Type of Solution</u>                               |
|-----------------|--------------|-------------------------|---|
| 1               | 38           | o                       | Analytic Solution                                     |
| 2               | 55           | x                       | Precessing Ellipse                                    |
| 3               | 63           | □                       | Numerical Solution<br>of Total Equations<br>of Motion |
| 4               | 44           | *                       | Initial Osculating<br>Ellipse                         |
| 5               | 42           | .                       | Numerical Residual<br>Perturbation Solu-<br>tion      |

n

Point number index in explanation of Runge-Kutta method in Section 3.4

|                     |   |                        |
|---------------------|---|------------------------|
| NDIV                | FORTRAN symbol for the error code   |                        |
|                     | <u>NDIV</u>   | <u>Reason for Halt</u> |
|                     | 1   | $a \leq 0$             |
|                     | 2   | $1 - e^2 \leq 0$       |
|                     | 7   | $S = 0$                |
|                     | 8   | $dt/d\psi = 0$         |
| NX                  | FORTRAN symbol; in subroutine PLOT, the number of characters in horizontal labels |                        |
| NY                  | FORTRAN symbol; in subroutine PLOT, the number of characters in vertical labels   |                        |
| OMEG1               | FORTRAN symbol for $\omega_1$   |                        |
| p                   | Subscript for numerical residual perturbation solution results                    |                        |
| P <sub>ss</sub>     | Period of mean precessing ellipse   |                        |
| P <sub>t</sub>      | A 700-element array of plot times   |                        |
| P <sub>t(ITP)</sub> | ITPth element of array P <sub>t</sub>   |                        |
| p <sub>1</sub>      | $\frac{1}{1 + \epsilon_2 \omega_1} - \epsilon_1 a^{3/2} (1 + \epsilon_1 m_1)$     |                        |
| p <sub>3</sub>      | $\frac{1}{1 + \epsilon_2 \omega_1} - \epsilon_a [2 + \epsilon(v_1 - 2\lambda_1)]$ |                        |
| P <sub>Ω</sub>      | Period of node  |                        |
| P <sub>w</sub>      | Period of perigee   |                        |

|       |   |
|-------|---|
| P1    | FORTRAN symbol for $p_1$  |
| P3    | FORTRAN symbol for $p_3$  |
| PHI   | FORTRAN symbol for $\phi$   |
| pk    | Subscript for precessing ellipse values   |
| PNOD  | FORTRAN symbol for the period of the node   |
| PPER  | FORTRAN symbol for the period of the perigee  |
| P1PS  | FORTRAN symbol for $2p_1\psi$ where $0 \leq \psi < 360$ approximately   |
| PSI   | FORTRAN symbol for $\psi$   |
| PS(I) | ith guess on $\psi$ in solution of Kepler's equation<br>= array of 25 elements  |
| PSS   | FORTRAN symbol for the two-body period of the precessing ellipse which is the reference orbit of the numerical residual perturbation solution |
| PTIME | FORTRAN symbol for the 700-element array $P_t$  |
| Q     | FORTRAN symbol for $q$  |
| q     | $= [(x'_s + \frac{\Delta x'}{2})\Delta x' + (y'_s + \frac{\Delta y'}{2})\Delta y' + (z'_s + \frac{\Delta z'}{2})\Delta z']/\rho_s^2$          |
| r     | Distance from larger body   |
| $r_a$ | $\sqrt{1 + 2\epsilon^2(x'\cos\gamma + y'\sin\gamma) + \epsilon^2\rho^2}$ where<br>$\rho^2 = x'^2 + y'^2 + z'^2$                               |

|        |   |
|--------|---|
| $r_f$  | Distance from larger body, according to fixed ellipse                               |
| $r_o$  | Initial distance from larger body   |
| $r$    | $\sqrt{1 + \epsilon^4 \rho^2 + 2\epsilon^2(x \cos \epsilon t + y \sin \epsilon t)}$ |
| RA     | FORTRAN symbol for $r_a$  |
| RADI   | FORTRAN symbol for inclination, $i$ (radians)                                       |
| RCU    | FORTRAN symbol for $r^3$  |
| RF     | FORTRAN symbol for $r_f$  |
| RHO    | FORTRAN symbol for $\rho$   |
| RHOASQ | FORTRAN symbol for $\rho_a^2$   |
| RHOCU  | FORTRAN symbol for $\rho^3$   |
| RHOD   | FORTRAN symbol for $\dot{\rho} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$          |
| RHOF   | FORTRAN symbol for $\rho_f$   |
| RHOFSQ | FORTRAN symbol for $\rho_f^2$   |
| RHOS   | FORTRAN symbol for $\rho_s$ = two-body orbit radius                                 |
| RHOSCU | FORTRAN symbol for $\rho_s^3$   |
| RHOSQ  | FORTRAN symbol for $\rho^2$   |
| RKSIMP | FORTRAN symbol for the name of Runge-Kutta subroutine                               |

|                  |  |
|------------------|--|
| RK1X, RK1Y, RK1Z | FORTRAN symbol for the Runge-Kutta values of $\Delta t$    |
| RK2X, RK2Y, RK2Z | times the acceleration on successive passes of Runge-Kutta |
| RK3X, RK3Y, RK3Z |  |
| RNAUT            | FORTRAN symbol for $r_o$                                   |
| RSQ              | FORTRAN symbol for $r^2$                                   |
| S                | FORTRAN symbol for s                                       |
| s                | Reciprocal of radius vector, $\rho$                        |
| $s_E$            | $\sin E$   |
| $s_i$            | $\sin i$   |
| $s_{p_1}$        | $\sin (2p_1 - 1)\psi$                                      |
| $s_{p_1\psi}$    | $\sin (p_1\psi)$   |
| $s_v$            | $\sin v$   |
| $s_{\theta_1}$   | $\sin \theta_1$  |
| $s_{\phi_s}$     | $\sin (v + \omega)$  |
| $s_{\epsilon t}$ | $\sin (\epsilon t)$  |
| $s_\psi$         | $\sin \psi$  |
| $s_\omega$       | $\sin \omega$  |
| $s_\Omega$       | $\sin \Omega$  |

|        |  |
|--------|--|
| S(10)  | FORTRAN symbol for the 10-element array in which values of time, position, velocity, and acceleration are saved for ordinary Runge-Kutta use   |
| SALPHA | FORTRAN symbol for $\sin \alpha_1$   |
| SC11PS | FORTRAN symbol for $\sin (C_{11}\psi)$   |
| SEC    | FORTRAN symbol for $\sin E$  |
| SEPT   | FORTRAN symbol for $\sin \gamma = S_{et}$  |
| SI     | FORTRAN symbol for $\sin i$  |
| SIPER  | FORTRAN symbol for $\sin \omega_s$   |
| SIS    | FORTRAN symbol for $\sin i_s$  |
| SIV    | FORTRAN symbol for $\sin v_s$  |
| SNO    | FORTRAN symbol for $\sin \Omega$   |
| SNODE  | FORTRAN symbol for $\Omega_s$  |
| SP     | FORTRAN symbol for the 10-element array in which values of time, position, velocity, and acceleration are saved when print or plot occurs at other than a regular compute point during the numerical solution of the total equations of motion |
| SP1    | FORTRAN symbol for $\sin (2p_1 - 1)\psi$   |
| SP1PS  | FORTRAN symbol for $\sin (p_1\psi + \alpha)$   |

|        |   |                        |                           |
|--------|---|------------------------|---------------------------|
| SP1PST | FORTRAN symbol for $\sin(p_1\psi)$  |                        |                           |
| SP3    | FORTRAN symbol for $\sin(p_3\psi)$  |                        |                           |
| SPDT   | FORTRAN symbol for the saved value of computing interval when print or plot occurs at other than a regular compute point during the numerical solution of the total equations of motion   |                        |                           |
| SPER   | FORTRAN symbol for $\omega_s$   |                        |                           |
| SPS    | FORTRAN symbol for $\sin \psi$  |                        |                           |
| SS     | FORTRAN symbol in subroutine RKSIMP for the 17-element array in which values of time, position, velocity, and acceleration are saved for <ul style="list-style-type: none"> <li>1. in case computing interval selection fails</li> <li>2. Simpson's rule integration</li> </ul> |                        |                           |
|        | $ss_1 = t$  | $ss_7 = \dot{z}_h$     | $ss_{13} = t_\Omega$      |
|        | $ss_2 = x_h$  | $ss_8 = \ddot{x}_h$    | $ss_{14} = p_\omega$      |
|        | $ss_3 = y_h$  | $ss_9 = \ddot{y}_h$    | $ss_{15} = \theta_1$      |
|        | $ss_4 = z_h$  | $ss_{10} = \ddot{z}_h$ | $ss_{16} = \sin \theta_1$ |
|        | $ss_5 = \dot{x}_h$  | $ss_{11} = t_s$        | $ss_{17} = \gamma$        |
|        | $ss_6 = \dot{y}_h$  | $ss_{12} = t_\psi$     |                           |
| SSNO   | FORTRAN symbol for $\sin \Omega_s$  |                        |                           |
| STHET1 | FORTRAN symbol for $\sin \theta_1$ . Old value not removed to avoid unnecessary change in code  |                        |                           |

|                         |   |
|-------------------------|---|
| STHETA                  | FORTRAN symbol for $\sin \theta_1$  |
| STRL                    | FORTRAN symbol for $\sin(v + \omega_s)$   |
| T                       | FORTRAN symbol for time, t  |
| t                       | Time  |
| $t_{a32}$               | $t \cdot a^{3/2}$   |
| $T_{\text{den } i}$     | ith term in Encke's series<br>$f = 3[1 - \frac{5}{2}q + \frac{5 \cdot 7}{2 \cdot 3}q^2 - \frac{5 \cdot 7 \cdot 9}{2 \cdot 3 \cdot 4}q^3 + \dots]$ |
| $T_f$                   | Final time  |
| $t_i$                   | Integer part of $t_R$   |
| $T_{\text{num } i}$     | Numerator by which ith term in series for f must be multiplied  |
| $T_{\text{num } (i-1)}$ | Numerator by which the (i-1)th term in series for f must be multiplied  |
| $t_p$                   | Print and plot time during numerical residual perturbation solution   |
| $t_{pT}$                | Print time during numerical solution of total equations of motion   |
| $t_R$                   | Time interval plotted on each page  |
| $t_s$                   | Time, modulo the period of precessing ellipse   |

|            |   |
|------------|---|
| $t_0$      | Initial time  |
| $T_1$      | Period of $t$ expression defined in<br>$t(2\pi) = T_1 = a^{3/2} 2\pi + \frac{15}{4} \epsilon a^3 e \sin [(2p_1 - 1)2\pi] - \epsilon^2 a^{9/2} \frac{11}{8} \sin 4p_1 \pi$ |
| $t_\psi$   | Time since last passage of apocenter by analytic solution   |
| $t_\Omega$ | Time since the last passage of the node through zero  |
| $t_w$      | Time since the last time the perigee was zero   |
| TAS32      | FORTRAN symbol for $t_s/a_s^{3/2}$  |
| TDEN       | FORTRAN symbol for the 40-element array of the denominators by which the terms in series for $f$ must be divided  |
| TERM       | FORTRAN symbol for the 40-element array of succeeding terms in series for $f$   |
| TF         | FORTRAN symbol for the total flight time; input as a run stop condition   |
| THETAL     | FORTRAN symbol for $\theta_1$   |
| THETA2     | FORTRAN symbol for $\theta_2$   |
| TN         | FORTRAN symbol for "next time" = $t + \Delta t$   |
| TNOD       | FORTRAN symbol for $t_\Omega$   |

|        |   |
|--------|---|
| TNUM   | FORTRAN symbol for the 40-element array of numerators by which the terms in the series for $f$ must be multiplied |
| TP1    | FORTRAN symbol for the print or plot interval   |
| TPER   | FORTRAN symbol for $t_{\omega}$   |
| TPI    | FORTRAN symbol for the print or plot time   |
| TPRINT | FORTRAN symbol for $t_p$ and $t_{pt}$   |
| TPSI   | FORTRAN symbol for $t_{\psi}$   |
| TR     | FORTRAN symbol for $t_R$  |
| TRUL   | FORTRAN symbol for $v + \omega_s$   |
| TS     | FORTRAN symbol for $t_s$  |
| V      | FORTRAN symbol for true anomaly in two-body equations,<br>$v$   |
| v      | true anomaly, $v$   |
| VSQ    | FORTRAN symbol for $\dot{\rho}^2$   |
| VSSQ   | FORTRAN symbol for $\dot{v}_s^2$ at initial point   |
| W      | FORTRAN symbol for the 30-element input array consisting of $w_1$ through $w_{30}$                                |
| $w_1$  | Input a   |

$w_2$  Input e

$w_3$  Input i (degrees)

$w_4$  Input  $\epsilon^4 (\equiv \mu)$

$w_5$  Input FTD modifier of  $\Delta t$

$w_6$  Not used

$w_7$  Input run stop time

$w_8$  Input maximum allowable error in Runge-Kutta

$w_9$  Input multiplier on computing interval if error  
< minimum allowable. Must be > 1. If it is not, it  
is set = 1.5

$w_{10}$  Input initial computing interval. If zero, it is  
set = 0.005

$w_{11}$  Input maximum failures permitted by program in  
selection of computing interval

$w_{12}$  Input minimum allowable error in Runge-Kutta

$w_{13}$  Input flag for secular rates  
If  $w_{13} \leq 0$ , only initial and final points printed  
If  $w_{13} > 0$ , approximately every 1/4 radian of  $\psi$   
printed

WR FORTRAN symbol for the 30-element input reference  
run array

|   |  |
|---|--|
| X, Y, Z                                   | FORTRAN symbol for x, y, z   |
| XD, YD, ZD                                | $\dot{x}, \dot{y}, \dot{z}$  |
| X2D, Y2D, Z2D                             | $\ddot{x}, \ddot{y}, \ddot{z}$   |
|   | Total position, velocity, and acceleration components.<br>Coordinate system referenced to plane of planetary motion. |
| x'  | Inertial coordinate in plane of massive bodies, from smaller body away from larger                                   |
| $x'_a$                                    | Inertial coordinate from analytic solution   |
| $x''_a$                                   | Coordinate in orbital plane direction to the node  |
| $x'_f$                                    | Inertial coordinate from fixed ellipse   |
| $x_h, \dot{x}_h, \ddot{x}_h$              | In subroutine RKSIMP, dummy variables of position, velocity, and acceleration, respectively                          |
| $\dot{x}_h, \ddot{x}_h, \ddot{\dot{x}}_h$ |  |
| $x_L$                                     | Time for the left edge of each plot (in subroutine PLOT) = EM  |
| $x'_s$                                    | Inertial coordinate from precessing ellipse  |
| $x_T$                                     | Time at right edge of each plot = EN   |
| $x'_o$                                    | Initial inertial coordinate  |
| XD  | See X, Y, Z, for definition  |
| X2D                                       | See X, Y, Z, for definition  |

|                     |  |
|---------------------|--|
| XF                  | FORTRAN symbol for the $x'$ coordinate of position determined from the initial osculating ellipse                      |
| XL                  | FORTRAN symbol in subroutine PLOT for left-most value of abscissa  |
| XNAUT               | FORTRAN symbol for $x'_o$  |
| XP, YP, ZP          | FORTRAN symbol for $x', y', z'$  |
| XDP, YDP, ZDP       | $\dot{x}', \dot{y}', \dot{z}'$   |
| X2DP, Y2DP, Z2DP    | $\ddot{x}', \ddot{y}', \ddot{z}'$  |
|                     | Approximate position, velocity and acceleration components. Coordinate system referenced to plane of planetary motion. |
| X2P, Y2P, Z2P       | FORTRAN symbol for $x'', y'', z''$   |
| XD2P, YD2P, ZD2P    | $\dot{x}'', \dot{y}'', \dot{z}''$  |
| X2D2P, Y2D2P, Z2D2P | $\ddot{x}'', \ddot{y}'', \ddot{z}''$   |
|                     | Approximate position, velocity and acceleration components. Coordinate system referenced to orbit plane of satellite   |
| XS, YS, ZS          | FORTRAN symbol for $x'_s, y'_s, z'_s$  |
| XSD, YSD, ZSD       | FORTRAN symbol for $\dot{x}'_s, \dot{y}'_s, \dot{z}'_s$  |
| y'                  | Inertial coordinate in plane of massive bodies, perpendicular to $x'$ in direction of motion                           |
| y'_a                | Inertial position component from analytic solution   |
| y''_a               | Coordinate in satellite orbital plane from analytic solution   |

|         |  |
|---------|--|
| $y'_f$  | Inertial coordinate from initial osculating ellipse  |
| $y'_s$  | Inertial coordinate from precessing ellipse  |
| YACAF   | FORTRAN symbol for $\Delta C_f$ and $\Delta C_f/C_{init}$  |
| YACAPP  | FORTRAN symbol for $\Delta C_a$ and $\Delta C_a/C_{init}$  |
| YACAS   | FORTRAN symbol for $\Delta C_s/C_{init}$   |
| YACOBI  | FORTRAN symbol for Jacobi constant on first point<br>$\Delta C_E$ . Thereafter equals difference between instantaneous value and initial value = $\Delta C_E/C_{init}$ |
| YB      | FORTRAN symbol in subroutine PLOT for the bottom value of the scale  |
| YF      | FORTRAN symbol for $y'$ coordinate of position as derived from the initial osculating ellipse  |
| YFD     | FORTRAN symbol for $y'$ coordinate of the velocity derived from the initial osculating ellipse   |
| YT      | Top of ordinate in plots   |
| $z'$    | Inertial coordinate normal to plane of massive bodies, positive north  |
| $z'_a$  | Inertial coordinate from analytic solution   |
| $z''_a$ | Coordinate perpendicular to orbital plane from analytic solution   |
| $z'_f$  | Inertial coordinate from fixed ellipse   |

|              |   |
|--------------|---|
| $z'_s$       | Inertial coordinate from precessing ellipse   |
| ZF           | FORTRAN symbol for $z'$ coordinate of the position<br>as derived from the initial osculating ellipse                              |
| ZFD          | FORTRAN symbol for $z'$ coordinate of the velocity<br>derived from the initial osculating ellipse                                 |
| $\alpha_1$   | Part of $2p_1 \psi_{\text{tot}}$ corresponding to an integral<br>number of revolutions of $\psi = 2n\pi + 2p_1$ , modulo $2\pi$   |
| $\alpha_2$   | $\alpha_1 + 2\pi + 2p_1$ , modulo $2\pi = 2(n+1)\pi + 2p_1$ , modulo<br>$2\pi$  |
| $\beta$      | Part of $p_3 \cdot \psi_{\text{tot}}$ corresponding to an integral<br>number of revolutions of $\psi = 2p_3 n\pi$ , modulo $2\pi$ |
| $\gamma$     | $\epsilon t$ , modulo $2\pi$  |
| $\Delta C$   | Error in Jacobi constant  |
| $\Delta C_a$ | Error in Jacobi constant calculated from analytic<br>solution   |
| $\Delta C_c$ | Error in Jacobi constant calculated from numerical<br>solution of total equations of motion                                       |
| $\Delta C_E$ | Error in Jacobi constant calculated from the residual<br>perturbation numerical solution  |
| $\Delta C_F$ | Error in Jacobi constant calculated from initial<br>osculating ellipse  |
| $\Delta C_s$ | Error in Jacobi constant calculated from precessing<br>ellipse  |

|   |  |
|---|--|
| $(\Delta C/C)_{ai}$                     | Fractional error of Jacobi constant of analytic solution   |
| $(\Delta C/C)_{ci}$                     | Fractional error of Jacobi constant for numerical solution of total equations of motion  |
| $(\Delta C/C)_{ki}$                     | Fractional error of Jacobi constant from initial osculating ellipse  |
| $(\Delta C/C)_{pi}$                     | Fractional error in Jacobi constant from numerical residual perturbation solution  |
| $(\Delta C/C)_{pki}$                    | Fractional error in Jacobi constant from precessing Kepler ellipse   |
| $\Delta E$                              | Difference between successive guesses at $E$ in iterative solution of Kepler's equation  |
| $\Delta E_p$                            | Difference between successive guesses at $E$ in iterative solution of perturbed Kepler's equation  |
| $\Delta t$                              | Computing interval   |
| $\Delta t_p$                            | Time interval for print  |
| $\Delta t_{sp}$                         | In numerical solution of total differential equations, the saved normal computing interval when a special print point is being computed  |
| $\Delta x'$ , $\Delta y'$ , $\Delta z'$ | Increments in position, velocity, and acceleration obtained numerically from the residual perturbation equations. Coordinate system is referenced to plane of planetary motion |

|  |  |
|--|--|
| $\Delta \dot{x}$ , $\Delta \dot{y}$ , $\Delta \dot{z}$ | In subroutine RKSIMP, the accumulated change in $\dot{x}$ , $\dot{y}$ , and $\dot{z}$ over two computing intervals                                       |
| $\Delta \rho_a$  | Distance between analytic solution and numerical residual perturbation solution  |
| $\Delta \dot{\rho}_a$                                  | Magnitude of vector velocity difference between the analytic solution and the numerical residual perturbation solution                                   |
| $\Delta \rho_c$  | Distance between numerical solution of total differential equations and the numerical residual perturbation solution                                     |
| $\Delta \dot{\rho}_c$                                  | Magnitude of vector velocity difference between the numerical solution of the total equations of motion and the numerical residual perturbation solution |
| $\Delta \rho_F$  | Distance between initial osculating ellipse solution and the numerical residual perturbation solution  |
| $\Delta \dot{\rho}_F$                                  | Magnitude of vector velocity difference between the initial osculating ellipse solution and the numerical residual perturbation solution                 |
| $\Delta \rho_s$  | Distance between the precessing ellipse solution and the numerical residual perturbation solution  |
| $\Delta \dot{\rho}_s$                                  | Magnitude of vector velocity difference between the precessing ellipse and the numerical residual perturbation solution                                  |
| $\epsilon$   | $\mu^{1/4}$  |

$$\epsilon_a \cdot a^{3/2} = \epsilon a_{3/2}$$

$$\epsilon_1 = \mu^{1/4}$$

$$\epsilon_2^2 = \mu^{1/2}$$

$$\epsilon_4^4 = \mu$$

$\epsilon_4$  Mass ratio of sun to sun + planet =  $\mu$

$\theta_1$  Part of  $(2p_1-1) \psi_{tot}$  resulting from integer n rotations of  $\psi = 2\pi n (2p_1-1)$ , modulo  $2\pi$

$\theta_2$   $\theta_1 + 2\pi (2p_1-1)$ , modulo  $2\pi = 2(n+1)\pi(2p_1-1)$ , modulo  $2\pi$

$\lambda_1$  Angular rate of change of  $\Omega$

$\mu$  Mass ratio of smaller body to sum of smaller and larger bodies

$$v_1^{3/2} (11 + 24e)/12 = \dot{\phi}/\epsilon^2$$

$\rho$  Radial distance from smaller body

$\dot{\rho}$  Total velocity in blown up coordinates

$\rho_a$  Radial distance obtained from analytic solution

$\rho_s$  Radial distance obtained from precessing ellipse

$\phi$  Angle from node to satellite

$\phi_s$  Angular distance of the satellite from the node  
as obtained from precessing ellipse;  $\phi_s = v + \omega$

$\phi_1$  Integer part of  $\phi$  modulo  $2\pi = \left[ \frac{2n\pi}{1+\epsilon^2\omega_1^2} - \epsilon^2 v_1 t(2\pi n) \right]$ , modulo  $2\pi$

$\psi$  True anomaly measured from apogee

$\Omega$  Argument of the node in analytic solution

$\omega$  Argument of pericenter

$$\omega_1 = \frac{a^3(24e - 7)}{12}$$

## Section 3

### DEVELOPMENT OF EQUATIONS

#### 3.0 GENERAL

The purpose of this section is to describe the sources of the equations used in the program and to outline the derivation of the exact forms programmed. When an equation from a reference is first introduced, the equation number in the source is shown to the left of the equation. The equation numbers of the present document are to the right of the equations. The equations are not derived in the exact order they are used; however, the numbers of the subsections in the Equations in Order of Solution, Section 4, where the equation is used, are stated after each derivation. Section 4 presents the equation numbers of this section so that the cross reference is complete.

#### 3.1 EQUATIONS OF MOTION

The motion considered is that of a satellite of negligible mass in close proximity to the smaller of the two massive bodies in the restricted three-body problem, which may serve as a simplified model for an actual case like a lunar satellite in the earth-moon system.

The coordinate system used is centered at the smaller mass and rotates with the constant angular velocity of the two massive bodies around their center of mass. With the notation as in reference 1, the  $x^*$ ,  $y^*$ , and  $z^*$  rectangular system is oriented such that the  $x^*, y^*$  plane coincides with the plane of motion of the two massive bodies, the  $x^*$ -axis pointing away from the larger body, the  $y^*$ -axis  $90^\circ$  ahead in the direction of motion, and  $z^*$ -axis parallel to the rotation axis as to complete a right hand system.

In order to nondimensionalize the equations of motion, the following scales are used:

1. Unit of length = Distance between the two massive bodies
2. Unit of time = Period of rotation of the  $x^*, y^*, z^*$  system in inertial space, divided by  $2\pi$

3. Unit of mass = Total mass of the system.

The equations of motion of the satellite in these coordinates are given in reference 1 by:

$$(1) \frac{d^2 \vec{x}^*}{dt^2} = \text{grad} * \left[ \frac{1 - \mu}{\vec{x}^* + i} + \frac{\mu}{\vec{x}^*} \right] - 2k \times \frac{d\vec{x}^*}{dt} - \hat{k} \times \{ \hat{k} \times [\vec{x}^* + (1 - \mu)i] \} \quad (1)$$

where  $\vec{x}^* = (x^*, y^*, z^*)$  and  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  are the unit vectors in the  $x^*$ ,  $y^*$ , and  $z^*$  directions.

Since the motion of the satellite takes place in a close vicinity of the smaller body, the following "blown-up" coordinates are introduced:

$$x = \frac{x^*}{\mu^{1/2}}, y = \frac{y^*}{\mu^{1/2}}, z = \frac{z^*}{\mu^{1/2}}, t = \frac{t^*}{\mu^{1/4}} \quad (2)$$

(Reference 1, page 205, equation above (6), there in vector notation) where  $\mu$  is the mass ratio of the smaller to the total mass.

In order to avoid terms of order  $\mu^{1/4} = \epsilon$  in the equations of motion, another transformation is made to inertially oriented coordinates by multiplying the vector  $\vec{x} = (x, y, z)$  by the matrix  $\{R\}$  with:

$$(7c) \quad \{R\} = \begin{bmatrix} \cos \epsilon t & -\sin \epsilon t & 0 \\ \sin \epsilon t & \cos \epsilon t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

[Reference 1, page 206, equation (7c)]

then

$$(7b) \quad \{\vec{x}\}' = \{R\}\{\vec{x}\} \quad (4)$$

The primed system is therefore inertially oriented, but centered at the smaller body.

When the transformations (2) and (4) are performed on the equations of motion (1), the following equations result:

$$\frac{d^2 \vec{x}'}{dt^2} = \frac{\vec{x}'}{\rho^3} - \frac{(1 - \epsilon^4)\epsilon^2 \vec{x}'}{r^3} + (1 - \epsilon^4)(1 - \frac{1}{r^3}) \begin{bmatrix} \cos \epsilon t \\ \sin \epsilon t \\ 0 \end{bmatrix} \quad (5)$$

where

$$\vec{x}' = (x', y', z') \quad (6)$$

$$\rho^2 = x'^2 + y'^2 + z'^2 \quad (7)$$

$$r^2 = 1 + \epsilon^4 \rho^2 + 2\epsilon^2(x' \cos \epsilon t + y' \sin \epsilon t) \quad (8)$$

$$\epsilon = \mu^{1/4} \quad (9)$$

Equations (5), (6), and (7) are used in Section 4.25. Equations (8) and (9) are used in Sections 4.25, 4.19.2, 4.16.2, and 4.8.3.

Geometrically,  $\rho$  is the distance of the satellite from the smaller body and  $r$  the distance between satellite and larger body, both in blown-up coordinates.

### 3.2 THE JACOBI CONSTANT

In order to check the accuracy of the solutions, the Jacobian integral will be used. From reference 2, page 281, equation (7), one obtains:

$$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2 = x^2 + y^2 + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} + c \quad (10)$$

This is in rotating coordinates, centered at the center of mass;  $r_1$  and  $r_2$  are the distances of the satellite from the larger and the smaller primary, respectively. To transform equation (10) to blown-up coordinates, centered at the smaller body, the above variables are to be replaced as follows:

$$\begin{aligned} x &\rightarrow 1 - \mu + \epsilon^2 x \quad \frac{dx}{dt} \rightarrow \epsilon \dot{x} \quad r_1 \rightarrow r \\ y &\rightarrow \epsilon^2 y \quad \frac{dy}{dt} \rightarrow \epsilon \dot{y} \quad r_2 \rightarrow \epsilon^2 \rho \\ z &\rightarrow \epsilon^2 z \quad \frac{dz}{dt} \rightarrow \epsilon \dot{z} \quad \mu \rightarrow \epsilon^4 \end{aligned} \quad (11)$$

Then after applying the transformation (4), the Jacobi integral takes on the form:

$$\begin{aligned} \dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 + 2\epsilon(\dot{x}'y' - \dot{y}'x') - 2(1 - \epsilon^4)(x' \cos \epsilon t + y' \sin \epsilon t) \\ - \frac{2}{\rho} - \frac{2(1 - \epsilon^4)}{\epsilon^2 r} + \frac{(1 - \epsilon^4)^2}{\epsilon^2} = C' \end{aligned} \quad (12)$$

The last terms on the left side are numerically large. To avoid the loss of significant figures in the computation of the error in the Jacobi constant, the initial value of the Jacobi constant is subtracted in the following way:

$$\begin{aligned} \dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 + 2\epsilon(\dot{x}'y' - \dot{y}'x') - 2(1 - \epsilon^4)(x' \cos \epsilon t + y' \sin \epsilon t) \\ - \frac{2}{\rho} - \frac{2(1 - \epsilon^4)}{\epsilon^2 r} + \frac{(1 - \epsilon^4)^2}{\epsilon^2} - C_{\text{init}} = \Delta C \end{aligned} \quad (13a)$$

$$\text{where } C_{\text{init}} = -\dot{x}'^2(0) - \dot{y}'^2(0) - \dot{z}'^2(0) - 2\epsilon[\dot{x}'(0)y'(0) - \dot{y}'(0)x'(0)]$$

$$+ 2(1 - \epsilon^4)x'(0) + \frac{2}{\rho(0)} + \frac{2(1 - \epsilon^4)}{\epsilon^2 r(0)} - \frac{(1 - \epsilon^4)^2}{\epsilon^2} \quad (13b)$$

The quantity  $\Delta C$  is zero if the solution is exact. For an approximate solution,  $\Delta C$  serves as a test quantity on the accuracy of the approximation. Define:

$$C_{\text{part}} = +\dot{x}'^2(0) + \dot{y}'^2(0) + \dot{z}'^2(0) + 2\epsilon[\dot{x}'(0)y'(0) - \dot{y}'(0)x'(0)] \\ - 2(1 - \epsilon^4)x'(0) + \frac{2}{\rho(0)} \quad (14)$$

Using equation (8) and noting that with apocenter as the initial condition, then the difference of the  $\frac{1}{\epsilon^2}$  terms of (13a) is written as:

$$-\frac{2(1 - \epsilon^4)}{\epsilon^2 r} + \frac{2(1 - \epsilon^4)}{\epsilon^2 r_0} = \frac{2(1 - \epsilon^4)}{\epsilon^2} \frac{(r^2 - r_0^2)}{rr_0(r + r_0)} \\ = 2 \frac{1 - \epsilon^4}{\epsilon^2} \frac{1 + \epsilon^4 \rho^2 + 2\epsilon^2(x' \cos \epsilon t + y' \sin \epsilon t) - 1 - \epsilon^4 x_0'^2 - 2\epsilon^2 x'}{rr_0(r + r_0)} \\ = 2(1 - \epsilon^4) \frac{2(x' \cos \epsilon t + y' \sin \epsilon t - x') + \epsilon^2(\rho^2 - x_0'^2)}{rr_0(r + r_0)} \quad (15)$$

where  $r(0) = 1 + \epsilon^2 x_0' = r_0$  (\*)

Then the error in the Jacobi constant becomes:

$$\Delta C' = (\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2) + 2[\epsilon(\dot{x}'y' - \dot{y}'x')] - \frac{1}{\rho} \\ - (1 - \epsilon^4)(x' \cos \epsilon t + y' \sin \epsilon t) - C_{\text{part}} \\ + 2(1 - \epsilon^4) \frac{[2(x' \cos \epsilon t + y' \sin \epsilon t - x_0') + \epsilon^2(\rho^2 - x_0'^2)]}{rr_0(r + r_0)} \quad (16)$$

Equation (16) is used in Sections 4.17.15, 4.28.1, and 4.28.2.

### 3.3 SOLUTIONS

In the following, three different methods for the solution of equation (5) will be outlined:

(\*) See equations (8), (42), and (61)

1. An approximate, analytic solution as given in reference 1.
2. A numerical residual perturbation method which uses the Keplerian part plus the secular perturbations to order  $\epsilon^2$  of the analytic solution as a reference orbit.
3. Direct application of numerical integration to the total exact equations of motion.

### 3.3.1 Analytic Solution

The developments as given in reference 1 start from an approximate set of equations that result from equation (5) when terms having  $\epsilon^3$  or a higher power of  $\epsilon$  as a factor are neglected.

The approximate equations of motion are:

$$(8a) \quad \frac{d^2x'}{dt^2} = -\frac{x'}{\rho^3} + \frac{\epsilon^2 x'}{2} + \frac{3}{2} \epsilon^2 (x' \cos 2\epsilon t + y' \sin 2\epsilon t)$$

$$\frac{d^2y'}{dt^2} = -\frac{y'}{\rho^3} + \frac{\epsilon^2 y'}{2} + \frac{3}{2} \epsilon^2 (x' \sin 2\epsilon t - y' \cos 2\epsilon t) \quad (17)$$

$$\frac{d^2z'}{dt^2} = -\frac{z'}{\rho^3} - \epsilon^2 z'$$

These equations are identical with the vector equation (8a) of reference 1.

The solution of equation (17) is given in reference 1 in the following form:

$$(18a) \quad s(\psi) = s_o(\psi) + \epsilon^2 s_1(\psi, \epsilon) + O(\epsilon^3)$$

$$(18b) \quad t(\psi) = t_o(\psi) + \epsilon^2 t_1(\psi, \epsilon) + O(\epsilon^3) \quad (18)$$

$$(18c) \quad z_a''(\psi) = z_0(\psi) + \epsilon^2 z_1(\psi, \epsilon) + O(\epsilon^3)$$

where

$$(25a) \quad s_0(\psi) = \frac{1 - e \cos \psi}{a(1 - e^2)} \quad (19)$$

$$(25b) \quad t_0(\psi) = a^{3/2} \left[ \frac{e(1 - e^2)^{1/2} \sin \psi}{1 - e \cos \psi} + \cos^{-1} \frac{\cos \psi - e}{1 - e \cos \psi} \right] \quad (20)$$

$$z_0(\psi) = 0 \quad (21)$$

$$(38) \quad s_1/a^2 = \frac{2}{3} (1 + 5e) - \frac{1}{3} [5 + (95e/8)] \cos \psi + \cos 2p_1 \psi$$

$$+ (5/8)e \cos (2p_1 + 1)\psi - 15qe \sin p_1 \psi$$

$$x \sin (p_1 - 1)\psi + O(e^2)$$

$$(39) \quad t_1/a^{9/2} = [(10/3) + (83e/12)] \sin \psi - (11/8) \sin 2p_1 \psi$$

$$+ (3e/4) \sin (2p_1 - 1)\psi - 2e \sin (2p_1 + 1)\psi$$

$$+ (5e/2) \sin 2\psi - 30qe \cos p_1 \psi$$

$$x \sin (p_1 - 1)\psi + O(e^2)$$

$$(42) \quad z_1 = (3ia^{4/64}) [\epsilon a^{3/2}]^{-1} \cos \frac{1}{2} (p_3 + 1)\psi \sin \frac{1}{2} (p_3 - 1)\psi$$

$$+ \sin \frac{1}{2} (p_3 + 1)\psi \cos \frac{1}{2} (p_3 - 1)\psi]$$

$$Q = - \left[ \frac{1}{4\epsilon a^{3/2}} + \frac{67}{80} + O(\epsilon) \right]$$

Equation (19) is used in Section 4.17.3.

The geometric interpretation of the variables in equation (18) is as follows:

$s$  = reciprocal radius vector from the smaller body to the satellite, in the blown-up scale.

$t$  = time, in the blown-up scale.

$z_a''$  = distance of the satellite from a plane that passes through the smaller body and is oriented with an inclination  $i$  and a node  $\Omega$  against the  $x',y'$  plane. The node is varying with time according to:

$$(12) \quad \Omega = \Omega_0 + \epsilon^2 \lambda_1 t \quad (39)$$

In order to restrict  $\Omega$  to values smaller than  $2\pi$ , equation (39) is written as:

$$\Omega = \Omega_0 + \epsilon^2 \lambda_1 T_\Omega \quad (39a)$$

used in Section 4.17.5, where  $T_\Omega$  is reduced in Section 4.17.1 by:

$$P_\Omega = \frac{2\pi}{\epsilon^2 \lambda} \quad (39b)$$

whenever it exceeds  $P_\Omega$ .  $\Omega_0$  is zero by the special choice of initial conditions in reference 1.

The constants  $a$ ,  $e$ , and  $i$  are the semi-major axis, eccentricity, and inclination of a Keplerian ellipse represented by the leading terms  $s_0$ ,  $t_0$ , and  $z_0$ . The angular variable,  $\psi$ , represents the true anomaly, but counted from the apocenter. All the terms of order  $\epsilon^2 e$  were dropped from  $s_1$ ,  $t_1$ ,

and  $z_1$  in this program for consistency since  $\epsilon$ ,  $e$ , and  $i$  were all considered small variables in reference 1 and the paper omitted  $\epsilon^3$  and  $\epsilon ei$  terms. Therefore, the equations for the above quantities become:

$$s_1(\psi, \epsilon) = a^2 \left[ \frac{2}{3} - \frac{5}{3} \cos \psi + \cos 2p_1 \psi + \frac{15}{8} \frac{e}{\epsilon a^{3/2}} (\cos \psi - \cos (2p_1 - 1)\psi) \right] \quad (22)$$

$$t_1(\psi, \epsilon) = a^{9/2} \left[ \frac{10}{3} \sin \psi - \frac{11}{8} \sin 2p_1 \psi - \frac{15}{4} \frac{e}{\epsilon a^{3/2}} (\sin \psi - \sin (2p_1 - 1)\psi) \right] \quad (23)$$

$$z_1(\psi, \epsilon) = \frac{3}{8} i \frac{a^{5/2}}{\epsilon(1 - \epsilon a^{3/2})} [\sin p_3 \psi - p_3 \sin \psi] \quad (24)*$$

Equation (22) is used in Section 4.17.3; equation (24) in Section 4.17.4.  
The eccentric anomaly:

$$E = \cos^{-1} \left( \frac{\cos \psi - e}{1 - e \cos \psi} \right) = \sin^{-1} \left( \frac{\sqrt{1 - e^2} \sin \psi}{1 - e \cos \psi} \right) \quad (25)$$

is defined such that

$$2n\pi \leq E \leq 2(n+1)\pi$$

for

$$2n\pi \leq \psi \leq 2(n+1)\pi$$

The time  $t$  and the eccentric anomaly are therefore single valued and increasing function of  $\psi$ . However, in order to avoid large arguments of

---

\*Equation (24) was taken from equation (57) of reference 7 instead of being taken from equation (42) of reference 1.

the trigonometric functions for large times, a time,  $t_\psi$ , which is the time since the last passage of the apocenter, is introduced. If the total  $\psi$  and the total  $E$  are denoted by  $\psi_{tot}$  and  $E_{tot}$ , and the symbols  $\psi$  and  $E$  are used for the principal values only, such that:

$$\psi_{tot} = 2\pi n + \psi$$

$$E_{tot} = 2\pi n + E$$

Then the equation for the time is written [c.f. equations (20), (23), and (25)]:

$$\begin{aligned} t(\psi_{tot}) &= a^{3/2}[E_{tot} + e \sin E_{tot}] - \frac{15}{4} \epsilon a^3 e [\sin \psi_{tot} - \sin (2p_1 - 1)\psi_{tot}] \\ &\quad + \epsilon^2 a^{9/2} [\frac{10}{3} \sin \psi_{tot} - \frac{11}{8} \sin 2p_1 \psi_{tot}] \end{aligned} \quad (26)$$

The time after  $n$  revolutions is:

$$\begin{aligned} t(2n\pi) &= a^{3/2} \cdot 2n\pi + \frac{15}{4} \epsilon a^3 e \sin [(2p_1 - 1) \cdot 2n\pi] \\ &\quad - \epsilon^2 a^{9/2} \cdot \frac{11}{8} \sin (2p_1 \cdot 2n\pi) \end{aligned} \quad (27)$$

Then, by subtraction of equation (27) from equation (26) and rearranging:

$$\begin{aligned} t(\psi_{tot}) - t(2n\pi) - t_\psi &= a^{3/2}[E_{tot} - 2\pi n + e \sin E_{tot}] \\ &\quad - \frac{15}{4} \epsilon a^3 e [\sin \psi_{tot} - \sin [(2p_1 - 1)(\psi_{tot} - 2\pi n)] \\ &\quad + 2\pi n (2p_1 - 1)] + \sin [(2p_1 - 1) \cdot 2n\pi] \\ &\quad + \epsilon^2 a^{9/2} [\frac{10}{3} \sin \psi_{tot} - \frac{11}{8} \sin [2p_1 (\psi_{tot} - 2\pi n)]] \end{aligned}$$

$$+ 4p_1 \pi n] + \frac{11}{8} \sin [4p_1 n \pi] \}$$

or

$$\begin{aligned} t_\psi = & a^{3/2} [E + e \sin E] - \frac{15}{4} \epsilon a^3 e \{ \sin \psi - \sin (2p_1 - 1)\psi \cos \theta_1 \\ & + [1 - \cos (2p_1 - 1)\psi] \sin \theta_1 + \epsilon^2 a^{9/2} \{ \frac{10}{3} \sin \psi \\ & - \frac{11}{8} \sin 2p_1 \psi \cos \alpha_1 + \frac{11}{8} (1 - \cos 2p_1 \psi) \sin \alpha_1 \} \end{aligned} \quad (28)$$

where

$$\theta_1 = [2n\pi(2p_1 - 1)] \text{ modulo } 2\pi$$

and

$$\alpha_1 = [2n\pi p_1] \text{ modulo } 2\pi$$

When  $n$  becomes large, forming the angle and then computing the modulo with respect to  $2\pi$  does not improve the accuracy. It is necessary to form the  $\theta_1$  and  $\alpha_1$  without developing large angles. For this purpose, initially  $\theta_1$  and  $\alpha_1$  are set to zero. The period of the first revolution  $T_{10}$  is computed by substituting  $\psi = 2\pi$  in equation (26) yielding:

$$t(2\pi) = T_1 = a^{3/2} 2\pi + \frac{15}{4} \epsilon a^3 e \sin [(2p_1 - 1)2\pi] - \epsilon^2 a^{9/2} \frac{11}{8} \sin 4p_1 \pi \quad (28a)$$

When  $t_\psi$  exceeds  $T_1$ , we set:

$$t_\psi = t_\psi - T_1 \quad (28b)$$

$$\theta_1 = [\theta_1 + 2\pi(2p_1 - 1)] \text{ modulo } 2\pi \quad (28c)$$

$$\alpha_1 = \{[(\alpha_1 + 2\pi p_1) \text{ modulo } 2\pi] + 2\pi p_1\} \text{ modulo } 2\pi \quad (28d)$$

The period for the next revolution is computed by applying equation (26) to obtain:

$$T_1 = t(2(n+1)\pi) - t(2n\pi)$$

or

$$T_1 = 2\pi a^{3/2} + \frac{15}{4} \epsilon a^3 e \{\sin \theta_2 - \sin \theta_1\} - \frac{11}{8} \epsilon^2 a^{9/2} \{\sin \alpha_2 - \sin \alpha_1\} \quad (28e)$$

where  $\theta_2$  is computed from the  $\theta_1$  just obtained by equation (28c) by:

$$\theta_2 = [\theta_1 + 2\pi(2p_1 - 1)] \text{ modulo } 2\pi \quad (28f)$$

and  $\alpha_2$  from the  $\alpha_1$  of equation (28d) by:

$$\alpha_2 = \{[(\alpha_1 + 2\pi p_1) \text{ modulo } 2\pi] + 2\pi p_1\} \text{ modulo } 2\pi \quad (28g)$$

Similarly the argument  $p_3\psi$  in  $z_a''$  [equation (24)] is taken as  $p_3\psi + \beta$ .  $\beta$  is set equal to zero initially and incremented each time  $t_\psi > T_1$  by:

$$\beta = (\beta + 2p_3\pi) \text{ modulo } 2\pi \quad (29)$$

Equations 28b through 29 are used in Section 4.17.1.2.

Now expression (28) may be solved for  $\psi$  when  $n$  and  $t_\psi$  are known. The resulting value is between 0 and  $2\pi$ . It is solved by Newton's method using  $M = \psi$  as a first guess in Section 4.17.2.4 with  $E$ , defined by equation (20), computed in Section 4.17.2.3.

The constants  $p_1$  and  $p_3$  are given by equations (27a) and (27c) of reference 1. However, these quantities can be obtained with more accuracy from (26a) and (26c) of reference 1,

by requiring:

$$\begin{aligned}\epsilon^2 \sin 2(\bar{\phi} - qt) &= \epsilon^2 \sin 2(p_1 \psi) + O(\epsilon^3) \\ \epsilon^2 \cos 2(\bar{\phi} - qt) &= \epsilon^2 \cos 2(p_1 \psi) + O(\epsilon^3)\end{aligned}\quad (29a)$$

$$\epsilon^2 \sin(\bar{\phi} - \ell t) = \epsilon^2 \sin(p_3 \psi) + O(\epsilon^3) \quad (29b)$$

$$\bar{\phi} - qt = p_1 \psi \quad (30)$$

$$\bar{\phi} - \ell t = p_3 \psi \quad (31)$$

where

$$(16a) \qquad q = \epsilon[1 + \epsilon m_1] \quad (32)$$

$$(16b) \qquad \ell = \epsilon[2 + \epsilon(v_1 - 2\lambda_1)] \quad (33)$$

$$(16c) \qquad m_1 = v_1 - \lambda_1 \quad (34)$$

$$\text{past (42)} \qquad v_1 = -\frac{1}{12} a^{3/2}(11 + 24e) \quad (35)$$

$$\text{past (41)} \qquad \lambda_1 = -\frac{3}{4} a^{3/2} \quad (36)$$

With the relations between  $\psi$  and  $\bar{\phi}$  given in equation (40) below and observing only the secular part of the relation between  $t$  and  $\psi$  in

equations (20) and (23) (see also equation (71) below), which gives  $t = a^{3/2}$ , the following values for  $p_1$  and  $p_3$  are derived without expansion:

$$p_1 = \frac{1}{1 + \epsilon^2 \omega_1^2} - \epsilon a^{3/2} (1 + \epsilon m_1) \quad (37)$$

$$p_3 = \frac{1}{1 + \epsilon^2 \omega_1^2} - \epsilon a^{3/2} [2 + \epsilon (v_1 - 2\lambda_1)] \quad (38)$$

Equations (34) through (38) are used in Section 4.4. The relation between  $s, t, z''$  and the primed coordinates is given by the following transformations of variables:

$$(17) \quad \psi = (1 + \epsilon^2 \omega_1^2) \bar{\phi} \quad (40)$$

$$(14) \quad \bar{\phi} = \phi + \epsilon^2 v_1 t \quad (41)$$

$$(13a) \quad s = \frac{1}{\rho} \quad (42)$$

$$(13b) \quad x_a'' = \rho \cos \phi \quad (43)$$

$$(13c) \quad y_a'' = \rho \sin \phi \quad (44)$$

$$(11a-11c) \quad \begin{bmatrix} x_a'' \\ y_a'' \\ z_a'' \end{bmatrix} = \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega \cos i & \cos \Omega \cos i & \sin i \\ \sin \Omega \sin i & -\cos \Omega \sin i & \cos i \end{bmatrix} \begin{bmatrix} x_a' \\ y_a' \\ z_a' \end{bmatrix} \quad (45)$$

$$\text{past (32)} \quad \omega = \omega_1 a^{-3} \quad (46a)$$

$$(36) \quad \omega = (-7 + 24e)/12 \quad (46b)$$

The initial conditions for any of the variables follow simply by putting  $\psi = 0$  for  $t = 0$ , which means that the motion starts at apocenter, which, in turn, coincides with the node for  $\Omega_0 = 0$ .

Equations (40) and (41) may be combined to give:

$$\phi = \frac{\psi}{1 + \epsilon^2 \omega_1^2} - \epsilon^2 v_1 t_\psi \quad (47)$$

which is used in Section 4.17.1.2.

To avoid large arguments of trigonometric functions in equations (43) and (44), these equations are written as:

$$x_a'' = \frac{1}{s} \cos (\phi + \phi_1) \quad (48)$$

$$y_a'' = \frac{1}{s} \sin (\phi + \phi_1)$$

where  $\phi_1$  is the value of  $\phi$  at  $n$  revolutions, modulo  $2\pi$ :

$$\phi_1 = \left( \frac{2n}{1 + \epsilon^2 \omega_1^2} - \epsilon^2 v_1 t(2n\pi) \right) \text{ modulo } 2\pi \quad (49)$$

Equation (48) is used in Section 4.17.4. As in the case of  $a_1$ ,  $\theta_1$ , and  $\beta$ ,  $\phi_1$  is formed in Section 4.17.1.2 at appropriate times by adding increments to the initial value  $\phi_1 = 0$  formed by evaluating equation (49) for  $n = 1$ . This avoids the angle ever becoming large, which would result in loss of accuracy.

Solving the linear system (45) for  $x'$ ,  $y'$ , and  $z'$  yields:

$$\left. \begin{aligned} x'_a &= x''_a \cos \Omega - y''_a \sin \Omega \cos i + z''_a \sin \Omega \sin i \\ y'_a &= x''_a \sin \Omega + y''_a \cos \Omega \cos i - z''_a \cos \Omega \sin i \\ z'_a &= y''_a \sin i + z''_a \cos i \end{aligned} \right\} \quad (50)$$

which are used in Section 4.17.6.

Differentiation of equations (18) through (24) with respect to  $\psi$  yields velocity components:

$$\begin{aligned} \frac{ds}{d\psi} &= \frac{e \sin \psi}{a(1 - e^2)} + \epsilon^2 a^2 \left( \frac{5}{3} \sin \psi - 2 \sin 2p_1 \psi \right. \\ &\quad \left. - \frac{15}{8} \epsilon a^{1/2} e [\sin \psi - \sin (2p_1 - 1)\psi] \right) \quad (51) \\ \frac{dt}{d\psi} &= \frac{a^{3/2}(1 - e^2)^{3/2}}{(1 - e \cos \psi)^2} - \frac{15}{4} \epsilon e a^3 [\cos \psi - \cos (2p_1 - 1)\psi \cos \theta \\ &\quad + \sin (2p_1 - 1)\psi \sin \theta] + \epsilon^2 a^{9/2} \left[ \frac{10}{3} \cos \psi - \frac{11}{4} \cos 2p_1 \psi \cos \alpha \right. \\ &\quad \left. + \frac{11}{4} \sin 2p_1 \psi \sin \alpha \right] \quad (52) \end{aligned}$$

$$\frac{dz''_a}{d\psi} = \frac{3}{8} \frac{i \epsilon a^{5/2}}{(1 - \epsilon a)^{3/2}} [\cos p_3 \psi - \cos \psi] \quad (53)$$

Terms of order  $\epsilon^3$ ,  $\epsilon^2 e$ ,  $\epsilon^3 i$ , and  $\epsilon e i$ , are neglected in equation (51) through (53). Equation (51) is used in Section 4.17.9 and equation (52) is used in Sections 4.17.7 and 4.17.2.5. Differentiation of (48) with respect to  $t$  yields

$$\begin{aligned} \dot{x}''_a &= - \frac{1}{s^2} \frac{ds}{dt} \cos (\phi - \phi_1) - \frac{1}{s} \frac{d\phi}{dt} \sin (\phi - \phi_1) \quad (53a) \\ \dot{y}''_a &= - \frac{1}{s^2} \frac{ds}{dt} \sin (\phi - \phi_1) + \frac{1}{s} \frac{d\phi}{dt} \cos (\phi - \phi_1) \end{aligned}$$

or

$$\dot{x}_a'' = -y_a'' \frac{d\phi}{dt} - \frac{x_a''}{s} \frac{ds}{dt} \quad (54a)$$

$$\dot{y}_a'' = +x_a'' \frac{d\phi}{dt} - \frac{y_a''}{s} \frac{ds}{dt} \quad (54b)$$

and from equation (53):

$$\dot{z}_a'' = \frac{3}{8} \frac{i\epsilon a^{5/2}}{(1-\epsilon a^{3/2})} [\cos p_3 \psi - \cos \psi] \frac{d\psi}{dt} \quad (54c)$$

where, from equation (47):

$$\frac{d\phi}{dt} = \frac{1}{1 + \epsilon^2 \omega_1^2} \frac{d\psi}{dt} - \epsilon^2 v_1 \quad (55)$$

and

$$\frac{d\psi}{dt} = \frac{1}{\frac{dt}{d\psi}}, \frac{ds}{dt} = \frac{ds}{d\psi} \cdot \frac{d\psi}{dt} \quad (56)$$

Equations (54a), (54b) are used in Section 4.17.10. Equations (55) and (56) are used in Section 4.17.9.

Differentiation of (50) with respect to time yields the velocity components in the inertial frame:

$$\left. \begin{aligned} \dot{x}'_a &= (-x_a'' \sin \Omega - y_a'' \cos \Omega \cos i + z_a'' \cos \Omega \sin i) \dot{\Omega} \\ &\quad + \dot{x}_a'' \cos \Omega - \dot{y}_a'' \sin \Omega \cos i + \dot{z}_a'' \sin \Omega \sin i \\ \dot{y}'_a &= (x_a'' \cos \Omega - y_a'' \sin \Omega \cos i + z_a'' \sin \Omega \sin i) \dot{\Omega} \\ &\quad + \dot{x}_a'' \sin \Omega + \dot{y}_a'' \cos \Omega \cos i - \dot{z}_a'' \cos \Omega \sin i \\ \dot{z}'_a &= \dot{y}_a'' \sin i + \dot{z}_a'' \cos i \end{aligned} \right\} \quad (57)$$

which are computed in Section 4.17.12.

### 3.3.1.1 Initial Conditions for the Analytic Solution

The initial conditions are obtained by setting  $\psi = t = \Omega = 0$ . From equations (19) through (24):

$$\begin{aligned}s_0(0) &= \frac{1}{a(1+e)} & s(0) &= 0 \\t_0(0) &= 0 & t_1(0) &= 0 \\z_0(0) &= 0 & z_1(0) &= 0\end{aligned}\tag{58}$$

From equations (51) through (53):

$$\frac{ds(0)}{d\psi} = 0\tag{59a}$$

$$\frac{dt(0)}{d\psi} = \frac{a^{3/2}(1+e)^{3/2}}{(1-e^2)^{1/2}} + \frac{7}{12} \epsilon^2 a^{9/2}\tag{59b}$$

$$\frac{dz''_a(0)}{d\psi} = 0\tag{59c}$$

Equation (59b) is used in Sections 4.8 and 4.19.2.

From equations (47), (48), and (98):

$$\phi(0) = 0\tag{60}$$

$$x''_a(0) = \frac{1}{s(0)} = a(1+e)\tag{61}$$

$$y''_a(0) = 0\tag{62}$$

$$z_a''(0) = 0 \quad (62a)$$

By (58) equations (61) and (62) are used in Section 4.8.3.

From equations (50) and (61):

$$\left. \begin{aligned} x_a'(0) &= x_a''(0) = a(1 + e) \\ y_a'(0) &= 0 \\ z_a'(0) &= 0 \end{aligned} \right\} \quad (63)$$

which are used in Section 4.8.3.

From (54a), (54b), (54c), (59a), (59c), and (62):

$$\left. \begin{aligned} \dot{x}_a''(0) &= 0 \\ \dot{y}_a''(0) &= x_a''(0) \cdot \frac{d\phi(0)}{dt} \\ \dot{z}_a''(0) &= 0 \end{aligned} \right\} \quad (64)$$

where by equation (55):

$$\frac{d\phi(0)}{dt} = [(1 + \epsilon^2 \omega_1^2) \frac{dt(0)}{d\psi}]^{-1} - \epsilon^2 v_1 \quad (65)$$

Equations (64) are computed in Sections 4.8.4 and equation (65) is used in Section 4.8.2 and 4.19.2.

And finally from (57), (61), (63), (62a), and (62):

$$\left. \begin{array}{l} \dot{x}'_a(0) = 0 \\ \dot{y}'_a(0) = x''_a(0) \cdot \Omega + \dot{y}''_a(0) \cos i \\ \dot{z}'_a(0) = \dot{y}''_a(0) \sin i \end{array} \right\} \quad (66)$$

which are computed in Sections 4.8.4 and 4.19.2.

### 3.3.1.2 Solution for Perturbed Kepler's Equation

Since the original equations of motion (5) are written with  $t$  being the independent variable, equation (28) has to be inserted to give  $\psi$  as a function of  $t$ . To accomplish this, the following iteration scheme is used to solve the perturbed Kepler's equation for  $E$ .

First guess of  $E$  for a given  $t_\psi$ :

$$E_1 = a^{-3/2} t_\psi - e \sin(t_\psi/a_{32}) (1 - e \cos t_\psi/a_{32}) \quad (67)$$

i.e., true anomaly = mean anomaly

Then use the formula

$$E_i = E_{i-1} + \frac{t_\psi - t_\psi(E_{i-1})}{\frac{dt(E_{i-1})}{dE}} \quad i = 2, 3, \dots \quad (68)$$

to obtain a better value for  $E$ , and so on. On the right side of equation (68), the solution (28) with the argument  $E_{i-1}$  is used. From equation (25)  $\psi$  is obtained as a function of  $E$  as:

$$\sin \psi = \sqrt{\frac{1 - e^2}{1 + e \cos E}} \sin E \quad (68aa)$$

$$\cos \psi = \frac{\cos E + e}{1 + e \cos E} \quad (68ab)$$

In the limit, the second term of equation (68) is to go to zero. However,  $t_\psi$  as defined by equation (28) is a very complex function. Round off may limit the smallness of this quantity. Taking the derivative of equation (28) in the form:

$$\frac{dt}{dE} = a^{3/2} (1 + e \cos E) + \epsilon^2 \frac{dt_1}{d\psi} \frac{d\psi}{dE} \quad (68ac)$$

where

$$\begin{aligned} \epsilon^2 \frac{dt_1}{d\psi} = & - \frac{15}{4} \epsilon a^3 e \{ \cos \psi - \cos [(2p_1 - 1)(\psi) + \theta_1] \} \\ & + \epsilon^2 a^{9/2} \left\{ \frac{10}{3} \cos \psi - \frac{11}{4} \cos (2p_1 \psi + \alpha_1) \right\} \end{aligned} \quad (68ad)$$

and with equation (68ab):

$$\epsilon^2 \frac{dt_1}{dE} = \epsilon^2 \frac{dt_1}{d\psi} \frac{d\psi}{dE} = \frac{\sqrt{1 - \epsilon^2}}{1 + e \cos E} \epsilon^2 \frac{dt_1}{d\psi} \quad (68ae)$$

and substituting in equation (68) gives after cancellation and rearrangement:

$$E_i = \frac{a^{3/2} e (E_{i-1} \cos E_{i-1} - \sin E_{i-1}) + E_{i-1} (\epsilon^2 \frac{dt_1}{dE})_{i-1} + t_\psi - t_{1(i-1)}}{a^{3/2} (1 + e \cos E) + \epsilon^2 \frac{dt_1}{dE}} \quad (68af)$$

where

$$\begin{aligned} t_1 = & - \frac{15}{4} \epsilon a^3 e \{ \sin \psi - \sin [(2p_1 - 1)\psi + \theta_1] + \sin \theta_1 \} \\ & + \epsilon^2 a^{9/2} \left\{ \frac{10}{3} \sin \psi - \frac{11}{8} [\sin (2p_1 \psi + \alpha_1) - \sin \alpha_1] \right\} \end{aligned} \quad (68ag)$$

Equations (68aa) and (68ab) are computed in Section 4.17.2.2; equations (68ad), (68ae), (68af), and (68ag), in Section 4.17.2.4.

When  $\psi$  is determined by arc tangent function, it is placed in the correct one of the first four quadrants by comparing the signs of the sine and cosine. However, when  $E$  is near 0 or  $2\pi$ , an intermediate step of the iteration may result in a value of  $E$  in the fifth or minus first quadrant. To assure that  $\psi$  and  $E$  be in the same quadrant when  $\cos \psi > 0$  the equation is then:

$$\psi = \text{sign } E[\psi + \pi(1 - 1 \text{ sign } \psi + 2 \text{ integer part of } \frac{E}{2\pi})] \quad (68ah)$$

### 3.3.1.3 Jacobi Integral

The test quantity,  $\Delta C_a$ , for the analytic solution is simply obtained by subscripting the coordinates in equation (16) with the subscript  $a$  as shown in Section 4.17.15.

The initial value of the Jacobi constant is given by [see equations (13b) and (14)]:

$$C_{\text{init}} = C_{\text{part}} - \frac{2(1 - \epsilon^4)}{\epsilon^2 r_o} \quad (68a)$$

Substitution of the initial conditions (63) and (66) in equation (14) yields:

$$\begin{aligned} C_{\text{part}} = & x_a''(0)^2 \dot{\Omega}^2 + 2x_a''(0)\dot{y}_a''(0)\dot{\Omega}\cos i + \dot{y}_a'(0)^2 \cos^2 i + \dot{y}_a''(0)^2 \sin^2 i \\ & - 2x_a''(0)[x_a''(0)\dot{\Omega} + \dot{y}_a''(0)\cos i] - 2x_a''(0)(1 - \epsilon^4) - \frac{2}{x_a''(0)} \end{aligned}$$

or

$$C_{\text{part}} = x'_a(0)^2 \{ \dot{\Omega}^2 + 2\dot{\Omega} \frac{d\phi(0)}{dt} \cos i + \left( \frac{d\phi(0)}{dt} \right)^2 \}$$

$$- 2\epsilon [\dot{\Omega} + \frac{d\phi(0)}{dt} \cos i] \} - 2(1 - \epsilon^4) x'_a(0) - \frac{2}{x'_a(0)} \quad (68b)$$

$C_{\text{init}}$  is computed in Section 4.8.6. The fractional error in the Jacobi constant is then given by  $\frac{\Delta C_a}{C_{\text{init}}}$

### 3.3.2 Numerical Residual Perturbation Method

The numerical residual perturbation method to be described below uses as a reference orbit the two-body ellipse as given by  $s_o, t_o$ , but accounts for the secular perturbations, which are reflected by the rotation of this ellipse with respect to the  $x', y', z'$  frame. This rotation is composed of the recession of the node as defined by equation (39) and the precession of the pericenter, which is comprised in equations (30) and (31). However, equations (30) and (31) contain some oscillatory terms due to the elliptic motion of the satellite. In order to separate the secular part, the argument of pericenter,  $\omega$ , is introduced as follows:

$$\phi = \psi + \omega \quad (69)$$

Substituting equations (52) and (41) in equation (40), one obtains:

$$\psi = \psi + \omega + \epsilon^2 \omega_1 \psi + \epsilon^2 \omega_1 \omega + \epsilon^2 v_1 t + \epsilon^4 \omega_1 v_1 t \quad (70)$$

Now from equation (26) or equation (27), it can be seen that the secular part of the relation between  $t$  and  $\psi$  is given by:

$$\psi = a^{-3/2} t \quad (71)$$

If equations (71), (46), and (35) are substituted in equation (70), one obtains for  $\omega$ :

$$\omega = \epsilon^2 \frac{\frac{3}{2} a^{3/2} - v_1 \epsilon^2 \omega_1}{1 + \epsilon^2 \omega_1^2} t_\omega = \dot{\omega} t_\omega \quad (72)$$

To keep  $\omega$  smaller than  $2\pi$ ,  $t$  is replaced by:

$$t_\omega = (t \text{ modulo } 2\pi)/\dot{\omega} \quad (72a)$$

The quantity  $\dot{\omega}$  is computed in Section 4.4,  $t_\omega$  is incremented in the RKSIMP subroutine (Section 7), and the modulo is formed in Section 4.17.1. The rotating ellipse is then described by:

$$\begin{aligned} \xi_s &= \rho_s \cos v \\ \eta_s &= \rho_s \sin v \\ \zeta_s &= 0 \end{aligned} \quad (73)$$

where the origin of  $\xi$ ,  $\eta$ ,  $\zeta$  is the smaller body,  $\xi$  points towards apocenter,  $\eta$  points  $90^\circ$  ahead of  $\xi$  in the direction of motion,  $\zeta$  is normal to the orbit plane, and

$$\begin{bmatrix} \xi_s \\ \eta_s \\ \zeta_s \end{bmatrix} = \{M\} \begin{bmatrix} x'_s \\ y'_s \\ z'_s \end{bmatrix} \quad (74)$$

The subscript  $s$  indicates approximate coordinates as derived from the reference orbit;  $\rho_s$  and  $v$  are radial distance and true anomaly, the latter measured from apocenter.

$\{M\}$  is a matrix composed of three rotation matrices: A rotation around the  $z'$  axis by the angle  $\Omega$ , described by (see page 41 ff. reference 3):

$$\{\Omega\} = \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (75)$$

A rotation around the line of nodes by the angle  $i$ , described by:

$$\{i\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \quad (76)$$

A rotation around the polar axis of the orbital plane by the angle  $\omega$ , described by:

$$\{\omega\} = \begin{bmatrix} \cos \omega & \sin \omega & 0 \\ -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (77)$$

Then

$$\{M\} = \{\omega\} \cdot \{i\} \{\Omega\} \quad (78)$$

$$\{M\} = \begin{bmatrix} \cos \omega \cos \Omega & \cos \omega \sin \Omega & \sin \omega \sin i \\ -\sin \omega \sin \Omega \cos i & +\sin \omega \cos \Omega \cos i & \cos \omega \sin i \\ -\sin \omega \cos \Omega & -\sin \omega \sin \Omega & \cos \omega \sin i \\ -\cos \omega \sin \Omega \cos i & +\cos \omega \cos \Omega \cos i & \cos \omega \sin i \\ +\sin \Omega \sin i & -\cos \Omega \sin i & \cos i \end{bmatrix} \quad (79)$$

Now, using the vector notation:

$$\vec{\xi}_s = \begin{bmatrix} \xi_s \\ \eta_s \\ \zeta_s \end{bmatrix} \quad \vec{x}'_s = \begin{bmatrix} x'_s \\ y'_s \\ z'_s \end{bmatrix} \quad (80)$$

in inertial coordinates the two-body solution may be written as:

$$\vec{x}'_s = \{M\}^T \vec{\xi}_s \quad (81)$$

or, using equation (73) and the transposed matrix of matrix (79)

$$\left. \begin{aligned} x'_s &= \rho_s [\cos(v + \omega) \cos \Omega - \sin(v + \omega) \sin \Omega \cos i] \\ y'_s &= \rho_s [\cos(v + \omega) \sin \Omega + \sin(v + \omega) \cos \Omega \cos i] \\ z'_s &= \rho_s \sin(v + \omega) \sin i \end{aligned} \right\} \quad (82)$$

Equation (82) is computed in Section 4.15.6.

The velocity components are obtained by differentiation of equation (81) with respect to time:

$$\dot{\vec{x}}'_s = \frac{d}{dt} \{M\}^T \cdot \vec{\xi}_s + \{M\}^T \dot{\vec{\xi}}_s \quad (83)$$

Differentiating again yields the accelerations:

$$\ddot{\vec{x}}'_s = \{M\}^T \ddot{\vec{\xi}}_s + 2 \frac{d\{M\}^T}{dt} \dot{\vec{\xi}}_s + \frac{d^2\{M\}^T}{dt^2} \vec{\xi}_s \quad (84)$$

Evaluating the matric derivatives yields:

$$\frac{d\{M\}^T}{dt} = -\dot{\Omega} \begin{bmatrix} \cos \omega \sin \Omega & -\sin \omega \sin \Omega & -\cos \Omega \sin i \\ +\sin \omega \cos \Omega \cos i & +\cos \omega \cos \Omega \cos i & \\ -\cos \omega \cos \Omega & \sin \omega \cos \Omega & -\sin \Omega \sin i \\ +\sin \omega \sin \Omega \cos i & +\cos \omega \sin \Omega \cos i & \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\ddot{\omega} \begin{bmatrix} \sin \omega \cos \Omega & \cos \omega \cos \Omega & 0 \\ +\cos \omega \sin \Omega \cos i & -\sin \omega \cos \Omega \cos i & \\ \sin \omega \sin \Omega & \cos \omega \sin \Omega & 0 \\ -\cos \omega \cos \Omega \cos i & +\sin \omega \cos \Omega \cos i & \\ -\cos \omega \sin i & \sin \omega \sin i & 0 \end{bmatrix} \quad (85)$$

$$\frac{d^2\{M\}^T}{dt^2} = -\dot{\Omega}^2 \begin{bmatrix} \cos \omega \cos \Omega & -\sin \omega \cos \Omega & +\sin \Omega \sin i \\ -\sin \omega \sin \Omega \cos i & -\cos \omega \sin \Omega \cos i & \\ \cos \omega \sin \Omega & -\sin \omega \sin \Omega & -\cos \Omega \sin i \\ +\sin \omega \cos \Omega \cos i & +\cos \omega \cos \Omega \cos i & \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ 2\ddot{\Omega}\dot{\omega} \begin{bmatrix} \sin \omega \sin \Omega & \cos \omega \sin \Omega & 0 \\ -\cos \omega \cos \Omega \cos i & +\sin \omega \cos \Omega \cos i & \\ -\sin \omega \cos \Omega & -\cos \omega \cos \Omega & 0 \\ -\cos \omega \sin \Omega \cos i & +\sin \omega \sin \Omega \cos i & \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\ddot{\omega}^2 \begin{bmatrix} \cos \omega \cos \Omega & -\sin \omega \cos \Omega & 0 \\ -\sin \omega \sin \Omega \cos i & -\cos \omega \sin \Omega \cos i & \\ \cos \omega \sin \Omega & -\sin \omega \sin \Omega & 0 \\ +\sin \omega \cos \Omega \cos i & +\cos \omega \cos \Omega \cos i & \\ \sin \omega \sin i & \cos \omega \sin i & 0 \end{bmatrix} \quad (86)$$

The rates of change of node and pericenter,  $\dot{\Omega}$  and  $\dot{\omega}$ , are obtained by differentiating equations (39) and (72) with respect to  $t$ :

$$\dot{\Omega} = -\epsilon^2 \lambda_1 \quad (87)$$

$$\dot{\omega} = \epsilon^2 \frac{\frac{3}{2} \frac{a^{3/2}}{2} - \epsilon^2 v_1 \omega_1}{1 + \epsilon^2 \omega_1^2} \quad (88)$$

Equations (87) and (88) are computed in Section 4.4.

The vector  $\dot{\xi}_s$  in Keplerian motion is given by:

$$\begin{aligned}\dot{\xi}_s &= -\frac{\sin v}{a^{1/2}(1-e^2)^{1/2}} \\ \dot{n}_s &= \frac{\cos v - e}{a^{1/2}(1-e^2)^{1/2}} \\ \dot{\zeta}_s &= 0\end{aligned}\quad (89)$$

Equations (89) can be derived from equation (73) by differentiation with respect to time, making use of the area-integral  $\rho_s^2 v = a^{1/2}(1-e^2)^{1/2}$  and the elliptic formula

$$\rho_s = \frac{a(1-e^2)}{1 - e \cos v} \quad (90)$$

Substituting equations (73), (89), (79), and (85) in equation (83) and rearranging yields the following expression for the approximate velocity vector:

$$\begin{aligned}\dot{x}'_s &= \frac{-1}{\sqrt{a(1-e^2)}} \{ + \sin(v+\omega) \cos \Omega + \cos(v+\omega) \sin \Omega \cos i \\ &\quad - e(\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos i) \} \\ &\quad - \rho_s \{ \cos(v+\omega) \sin \Omega [\dot{\Omega} + \dot{\omega} \cos i] \\ &\quad + \sin(v+\omega) \cos \Omega [\dot{\omega} + \dot{\Omega} \cos i] \}\end{aligned}\quad (91)$$

$$\begin{aligned}
\dot{\vec{y}}'_s &= \frac{1}{\sqrt{a(1-e^2)}} \{ -\sin \Omega \sin(v+\omega) + \cos \Omega \cos(v+\omega) \cos i \\
&\quad + e[\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos i] \} \\
&\quad + \rho_s \{ \cos(v+\omega) \cos \Omega (\dot{\Omega} + \dot{\omega} \cos i) \\
&\quad - \sin(v+\omega) \sin \Omega (\dot{\omega} + \dot{\Omega} \cos i) \}
\end{aligned} \tag{92}$$

$$\dot{\vec{z}}'_s = \frac{\sin i}{\sqrt{a(1-e^2)}} [\cos(v+\omega) - e \cos \omega] + \rho_s \dot{\omega} \sin i \cos(v+\omega) \tag{93}$$

Equations (91) through (93) are used in Section 4.17.13.

The residual perturbation equations of motion are obtained by substitution of the two-body equations of motion, i.e.:

$$\ddot{\vec{\xi}}_s = - \frac{\vec{\xi}_s}{\rho_s^3} \tag{94}$$

in equation (84), substitution of equation (74) in the result, and then subtraction of this from the exact restricted three-body equations (5), yielding:

$$\begin{aligned}
\ddot{\vec{x}}' - \ddot{\vec{x}}'_s &= \Delta \ddot{\vec{x}}' = - \frac{\dot{\vec{x}}'_s}{\rho_s^3} + \frac{\dot{\vec{x}}'_s}{\rho_s^3} - \frac{(1-\epsilon^4)\epsilon^2 \vec{x}'_s}{r^3} + (1-\epsilon^4)(1-\frac{1}{r^3}) \begin{bmatrix} \cos \epsilon t \\ \sin \epsilon t \\ 0 \end{bmatrix} \\
&\quad - 2 \frac{d\{M\}^T}{dt} \dot{\vec{\xi}}_s - \frac{d^2\{M\}^T}{dt^2} \vec{\xi}_s
\end{aligned} \tag{95}$$

The vector  $\Delta \vec{x}'$  is the correction vector which is to be added to the solution of the reference orbit in order to obtain the correct coordinates, i.e.,

$$\vec{x}' = \vec{x}'_s + \Delta\vec{x}' \quad (96)$$

(This addition is accomplished in Section 4.16.2.)

Equations (95) are to be integrated numerically. The first two terms on the right side of equation (95) represent a small difference of two large numbers. In order to avoid the loss of significant figures, the Encke transformation as described in detail in reference 3, starting at the bottom of page 176, will be applied.

Substitution of equations (85), (86), (89), and (73) in equation (95) and rearranging yields:

$$\begin{aligned} \ddot{\Delta x}' &= \frac{1}{\rho_s^3} [fq(x'_s + \Delta x') - x'] - \frac{(1 - \epsilon^4)\epsilon^2 x'}{r^3} + (1 - \epsilon^4)(1 - \frac{1}{r^3}) \cos \epsilon t \\ &+ \frac{2\dot{\Omega}}{\sqrt{a(1 - e^2)}} \{- \sin(v + \omega) \sin \Omega + \cos(v + \omega) \cos \Omega \cos i \\ &+ e (\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos i)\} \\ &+ \frac{2\dot{\omega}}{\sqrt{a(1 - e^2)}} \{+ \cos(v + \omega) \cos \Omega - \sin(v + \omega) \sin \Omega \cos i \\ &- e (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i)\} \\ &- 2\rho_s \dot{\Omega} \dot{\omega} \{\sin(v + \omega) \sin \Omega - \cos(v + \omega) \cos \Omega \cos i\} \\ &+ \rho_s (\dot{\omega}^2 + \dot{\Omega}^2) \{\cos(v + \omega) \cos \Omega - \sin(v + \omega) \sin \Omega \cos i\} \end{aligned} \quad (98)$$

$$\begin{aligned} \ddot{\Delta y}' &= \frac{1}{\rho_s^3} [fq(y'_s + \Delta y') - y'] - \frac{(1 - \epsilon^4)\epsilon^2(y'_s + \Delta y')}{r^3} + (1 - \epsilon^4)(1 - \frac{1}{r^3}) \sin \epsilon t \\ &+ \frac{2\dot{\Omega}}{\sqrt{a(1 - e^2)}} \{\sin(v + \omega) \cos \Omega + \cos(v + \omega) \sin \Omega \cos i \end{aligned}$$

$$\begin{aligned}
& - e (\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos i) \\
& + \frac{2\dot{\omega}}{\sqrt{a(1-e^2)}} \{ \cos(v+\omega) \sin \Omega + \sin(v+\omega) \cos \Omega \cos i \\
& - e (\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i) \\
& - 2\rho_s \ddot{\Omega} \{- \sin(v+\omega) \cos \Omega - \cos(v+\omega) \sin \Omega \cos i\} \\
& + \rho_s (\dot{\Omega}^2 + \dot{\omega}^2) \{ \cos(v+\omega) \sin \Omega + \sin(v+\omega) \cos \Omega \cos i \} \quad (99)
\end{aligned}$$

$$\begin{aligned}
\ddot{z}' &= \frac{1}{\rho_s^3} [f q(z'_s + \Delta z') - \frac{(1-\varepsilon^4)\varepsilon^2(z'_s + \Delta z')}{r^3}] \\
&+ \dot{\omega} \sin i \{ [\frac{2}{\sqrt{a(1-e^2)}} + \rho_s \dot{\omega}] \sin(v+\omega) - \frac{2e}{\sqrt{a(1-e^2)}} \sin \omega \} \\
& \quad (100)
\end{aligned}$$

Equations (98) through (100) are used in Section 4.16.2 where the quantities  $f$  and  $q$  are defined by:

$$q = \frac{1}{\rho_s^2} [(x'_s + \frac{1}{2} \Delta x') \Delta x' + (y'_s + \frac{1}{2} \Delta y') \Delta y' + (z'_s + \frac{1}{2} \Delta z') \Delta z'] \quad (101)$$

$$f = \frac{1 - (1 + 2q)^{-3/2}}{9} \quad (102)$$

$$\begin{aligned}
((7,6)) \quad &= 3[1 - \frac{5}{2}q + \frac{5 \cdot 7}{2 \cdot 3}q^2 - \frac{5 \cdot 7 \cdot 9}{2 \cdot 3 \cdot 4}q^3 + \dots] \\
&= 3\{1 - \frac{5}{2}q(1 - \frac{7}{3}q(1 - \frac{9}{4}q(1 - \frac{11}{5}q(\dots))))\}
\end{aligned}$$

The above expansion can be found in reference 4, page 98, equation ((7,6)). They are evaluated in Section 4.16.1.

Now it is still necessary to define  $\rho_s$  and  $v$  as functions of time. Introducing the eccentric anomaly  $E$ , the theory of elliptic motion (e.g., reference 2, page 164, equations (48) and (49), with  $k(\sqrt{l+m})$  being  $l$  in this case) gives:

$$t = a^{-3/2} [E + e \sin E] \quad (103)$$

$$\cos v = \frac{\cos E + e}{1 + e \cos E} \quad (104)$$

$$\sin v = \frac{\sqrt{1 - e^2} \sin E}{1 + e \cos E} \quad (105)$$

$$\rho_s = a(l + e \cos E) \quad (106)$$

There are plus signs instead of minus signs in the above equations because all the anomalies are measured from apocenter. Equations (104) and (105) are evaluated in Section 4.15.2. Equations (106) is evaluated in Section 4.15.3. Kepler's equation (103) is solved for  $E$  in Section 4.15.1 by iteration as follows:

$$\begin{aligned} E_{i+1} &= E_i + \frac{a^{-3/2} [t_s - t_s(E_i)]}{\frac{dt_s(E_i)}{dE}} = E_i + \frac{a^{-3/2} t_s - E_i - e_s \sin E_i}{1 - e_s \cos E_i} \\ &= \frac{a^{-3/2} t_s - e_s (E_i \cos E_i - \sin E_i)}{1 - e_s \cos E_i} \end{aligned} \quad (107)$$

with a first estimate:

$$E_1 = a^{-3/2} t_s - e_s \sin(a^{-3/2} t_s) + \frac{1}{2} e_s^2 \sin 2(a^{-3/2} t_s) \quad (108)$$

Equation (108) is taken from reference 2, page 169, equation (59);  $M$  being  $a^{-3/2} t_s$ . The quantity  $t_s$  is the input time, modulo  $2\pi a^{3/2}$  (see Section 17.1).

### 3.3.2.1 Initial Conditions for the Residual Perturbation Solution

The initial conditions on the residual perturbation equations of motion (98) through (100) are chosen such that for  $t = 0$  the total values of  $x'$ ,  $y'$ , and  $z'$  agree with the values  $x'_a$ ,  $y'_a$ , and  $z'_a$  of the analytic solution.

$$\vec{x}'(0) = \vec{x}'_s(0) + \Delta \vec{x}'(0) = \vec{x}'_a(0) \quad (109)$$

They are set equal in Section 4.8.3. The initial conditions for the reference orbit are:

from equations (103) through (106), (73), (39), (72), and (82):

$$E(0) = v(0) = 0, \rho_s(0) = a(1 + e) \quad (110)$$

$$\xi_s(0) = \rho_s(0), \eta_s(0) = \zeta_s(0) = 0 \quad (111)$$

$$\Omega(0) = 0 \quad (112)$$

$$\omega(0) = 0$$

$$x'_s(0) = \rho_s(0)$$

$$y'_s(0) = z'_s(0) = 0 \quad (113)$$

The initial conditions from equations (113) are established in Section 4.8.3

From equations (91) through (93):

$$\dot{x}'_s(0) = 0 \quad (114)$$

$$\begin{aligned} \dot{y}'_s(0) &= [a(1 - e^2)]^{-1/2} \cos i (1 - e) + \rho_s(0)(\dot{\Omega} + \dot{\omega} \cos i) \\ &= x'_s(0)[\dot{\Omega} + (\frac{1}{dt_o(0)} + \dot{\omega}) \cos i] \end{aligned} \quad (115)$$

$$\begin{aligned} \dot{z}'(0) &= \{[a(1 - e^2)]^{-1/2}(1 - e) + \rho_s(0) \dot{\omega}\} \sin i \\ &= x'_s(0)[\frac{1}{dt_o(0)} + \dot{\omega}] \sin i \end{aligned} \quad (116)$$

Equation (114) is computed in Section 4.8.6. Equations (115) and (116) are computed in Section 4.8.4.

Now from equation (109):

$$\begin{aligned} \Delta \vec{x}'(0) &= \vec{x}'_a(0) - \vec{x}'_s(0) \\ \dot{\Delta \vec{x}'}(0) &= \dot{\vec{x}}'_a(0) - \dot{\vec{x}}'_s(0) \end{aligned} \quad (117)$$

Substituting equations (113) through (116), (63), and (66) in equation (117) yields:

$$\left. \begin{array}{l} \Delta x'(0) = 0 \\ \Delta y'(0) = 0 \\ \Delta z'(0) = 0 \\ \dot{\Delta x}'(0) = 0 \end{array} \right\} \quad (118)$$

$$\Delta \dot{y}'(0) = x' \dot{G} \cos i$$

(119)

$$\Delta \dot{z}'(0) = x' \dot{G} \sin i$$

where

$$\begin{aligned} \dot{G} &= [(1 + \epsilon^2 \omega_1^2) \frac{dt(0)}{d\psi}]^{-1} - \epsilon^2 v_1 - \left(\frac{dt_o(0)}{d\psi}\right)^{-1} - \dot{\omega} \\ &= \frac{\frac{dt_o(0)}{d\psi} \left(\frac{dt_o(0)}{d\psi} + \epsilon^2 \frac{dt_1(0)}{d\psi}\right) (1 + \epsilon^2 \omega_1^2)}{\frac{dt(0)}{d\psi} (1 + \epsilon^2 \omega_1^2) \frac{dt_o(0)}{d\psi}} - \epsilon^2 v_1 - \dot{\omega} \\ &= \frac{-\epsilon^2 \omega_1^2 \frac{dt_o(0)}{d\psi} - \epsilon^2 \frac{dt_1(0)}{d\psi} (1 + \epsilon^2 \omega_1^2)}{\frac{dt(0)}{d\psi} \frac{dt_o(0)}{d\psi} (1 + \epsilon^2 \omega_1^2)} - \epsilon^2 v_1 - \dot{\omega} \end{aligned} \quad (120)$$

The  $\Delta$  velocities are not zero at the initial time because the precessing ellipse has been chosen to be a mean ellipse rather than an initial osculating ellipse. This was necessary in order that the period of the ellipse did not vary in a secular manner from the period of the actual motion. Equations (117) through (120) are computed in Section 4.8.4.

Substitution of the initial conditions into equations (98) through (100) yields the initial residual accelerations:

$$\begin{aligned} \ddot{\Delta x}'(0) &= \frac{1 - \epsilon^4}{r_o^3} [2\epsilon^2 x'(0) + 3\epsilon^4 x'(0)^2 + \epsilon^6 x'(0)^3] \\ &\quad + \frac{2(1 - e)}{\sqrt{a(1 - e^2)}} (\dot{\omega} \cos i + \dot{\omega}) \\ &\quad + 2x'(0)\dot{\omega} \cos i + x'(0)(\dot{\omega}^2 + \dot{\Omega}^2) \end{aligned} \quad (121)$$

$$\ddot{\Delta y}'(0) = \ddot{\Delta z}'(0) = 0 \quad (122)$$

which are computed in Section 4.8.5.

### 3.3.3 Fixed Reference Orbit

For comparison, a fixed, osculating Kepler ellipse is also calculated from the initial conditions. The node and the argument of apocenter are zero at  $t = 0$ , and the inclination has the same value as in the previous solutions. However, the osculating values for  $a$  and  $e$  are different from those used previously because the perturbations in the velocities do not vanish initially in the analytic solution.

To obtain the osculating values for  $a$ , the formula in reference 3, page 48, equation (88) is used:

$$\frac{1}{a} = \frac{2}{r} - v^2 \quad (123)$$

or, in our notation:

$$a = \frac{1}{\frac{2}{\rho(0)} - \dot{\rho}(0)^2} \quad (124)$$

On the same page, further below, the following equation is found:

$$e \cos u = 1 - \frac{r}{a} \quad (125)$$

Since the eccentric anomaly,  $u$ , is  $180^\circ$  initially, this becomes in our notation:

$$e = \frac{\rho(0)}{a} - 1 = \frac{x'(0)}{a} - 1 \quad (126)$$

Equations (124) and (126) are used in Section 4.19.3. The coordinates and velocity components are then obtained as in equations (82) and (91) through (93), but setting  $\omega = \Omega = \dot{\omega} = \dot{\Omega} = 0$ :

$$x'_f = \rho_s \cos v$$

$$y'_f = \rho_s \sin v \cos i$$

$$z'_f = \rho_s \sin v \sin i \quad (127)$$

$$\dot{x}'_f = - [a(1 - e^2)]^{-1/2} \sin v$$

$$\dot{y}'_f = + [a(1 - e^2)]^{-1/2} (\cos v - e) \cos i$$

$$\dot{z}'_f = + [a(1 - e^2)]^{-1/2} (\cos v - e) \sin i$$

Equations (127) are computed in Section 4.28.2.

Again, the Jacobi constant is calculated according to equation (16), subscripting all the variables with f, indicating that they are computed from the fixed orbit. The radius vector and the total velocity are also compared with the corresponding values obtained by the numerical residual perturbation method:

$$\Delta\rho_f = \frac{1}{a} [(x'_f - x')^2 + (y'_f - y')^2 + (z'_f - z')^2]^{1/2}$$
$$\Delta\dot{\rho}_f = a^{1/2} [(\dot{x}'_f - \dot{x}')^2 + (\dot{y}'_f - \dot{y}')^2 + (\dot{z}'_f - \dot{z}')^2]^{1/2} \quad (128)$$

Equations (128) are used in Section 4.28.2

The quantities  $\Delta\rho_f$  and  $\Delta\dot{\rho}_f$  are normalized with the semi-major axis and the circular velocity  $a^{-1/2}$ .

### 3.3.4 Numerical Integration of Total Equations

For the integration of the total equations the same initial conditions as in the analytic solution are used, but the exact equations of motion (5)

are integrated numerically. The results are checked by means of the Jacobi integral equation (16). The radius vector and the total velocity are also compared with the values obtained from the numerical residual perturbation method by computing the differences, normalized with  $a$  and  $a^{-1/2}$ , respectively:

$$\begin{aligned}\Delta \rho_c &= \frac{1}{a} [(x'_c - x')^2 + (y'_c - y')^2 + (z'_c - z')^2]^{1/2} \\ \Delta \dot{\rho}_c &= a^{1/2} [(\dot{x}'_c - \dot{x}')^2 + (\dot{y}'_c - \dot{y}')^2 + (\dot{z}'_c - \dot{z}')^2]^{1/2}\end{aligned}\quad (129)$$

where  $x'_c, y'_c, z'_c, \dot{x}'_c, \dot{y}'_c$ , and  $\dot{z}'_c$  stand for the values obtained by numerical integration of the exact equations of motion (5). Equations (129) were used in Section 4.28.2.

### 3.4 METHOD FOR NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

The Runge-Kutta method is used for the numerical solution of the differential equations. The method is a simple extension of the methods for the second order and simultaneous equations given by Hildebrand (reference 5, page 237) which are:

- Given the simultaneous first order equations:

$$(6.16.7) \quad \begin{aligned}\frac{dy}{dx} &= F(x,y,u), \\ \frac{du}{dx} &= G(x,y,u)\end{aligned}\quad (130)$$

$$\frac{du}{dx} = G(x,y,u)$$

the solution may be written as:

$$(6.16.8) \quad \begin{aligned}y_{n+1} &= y_n + \frac{1}{6} (k_0 + 2k_1 + 2k_2 + k_3) + O(h^5) \\ u_{n+1} &= u_n + \frac{1}{6} (m_0 + 2m_1 + 2m_2 + m_3) + O(h^5)\end{aligned}\quad (131)$$

where

$$(6.16.9) \quad k_0 = hF(x_n, y_n, u_n),$$

$$k_1 = hF(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0, u_n + \frac{1}{2}m_0),$$

$$k_2 = hF(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, u_n + \frac{1}{2}m_1), \quad (132)$$

$$k_3 = hF(x_n + h, y_n + k_2, u_n + m_2)$$

and

$$(6.16.10) \quad m_0 = hG(x_n, y_n, u_n),$$

$$m_1 = hG(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0, u_n + \frac{1}{2}m_0),$$

$$m_2 = hG(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, u_n + \frac{1}{2}m_1), \quad (133)$$

$$m_3 = hG(x_n + h, y_n + k_2, u_n + m_2).$$

## 2. The second order equation

$$(6.16.11) \quad \frac{d^2y}{dx^2} = G(x, y, y'), \quad (134)$$

can be written as the two simultaneous first order differential equations

$$\frac{dy}{dx} = u$$

and

$$\frac{du}{dx} = G(x, y, u)$$

Then equation (6.16.9) gives:

$$k_0 = hy'_n, \quad k_1 = hy'_n + \frac{h}{2} m_0, \quad k_2 = hy'_n + \frac{h}{2} m_1, \quad k_3 = hy'_n + hm_2,$$

and hence equations (6.16.8) and (6.16.10) give:

$$(6.16.12) \quad \left. \begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{h}{6} (m_0 + m_1 + m_2) + O(h^5), \\ y'_{n+1} &= y'_n + \frac{1}{6} (m_0 + 2m_1 + 2m_2 + m_3) + O(h^5), \end{aligned} \right\} \quad (135)$$

where

$$(6.16.13) \quad \left. \begin{aligned} m_0 &= hG(x_n, y_n, y'_n), \\ m_1 &= hG(x_n + \frac{1}{2} h, y_n + \frac{1}{2} hy'_n, y'_n + \frac{1}{2} m_0), \\ m_2 &= hG(x_n + \frac{1}{2} h, y_n + \frac{1}{2} hy'_n + \frac{1}{2} hm_0, y'_n + \frac{1}{2} m_1), \\ m_3 &= hG(x_n + h, y_n + hy'_n + \frac{1}{2} hm_1, y'_n + m_2). \end{aligned} \right\} \quad (136)$$

The extension of the above technique to three second-order differential equations:

$$\frac{d^2x_i}{dt^2} = f_i(t, x_1, x_2, x_3) \text{ where } i = 1, 2, 3 \quad (137)$$

yields

$$\frac{dx_i}{dt} = g_i(t, x_1, x_2, x_3) \text{ for } i = 1, 2, 3 \quad (138)$$

$$\frac{dg_i}{dt} = f_i(t, x_1, x_2, x_3) \quad (139)$$

$$m_{oi} = hf_i(t_n, x_{1n}, x_{2n}, x_{3n}) \quad (140)$$

$$m_{1i} = hf_i(t_n + \frac{h}{2}, x_{1n} + \frac{h}{2}\dot{x}_{1n}, x_{2n} + \frac{h}{2}\dot{x}_{2n}, x_{3n} + \frac{h}{2}\dot{x}_{3n}) \quad (141)$$

$$m_{2i} = hf_i(t_n + \frac{h}{2}, x_{1n} + \frac{h}{2}\dot{x}_{1n} + \frac{h}{4}m_{o1}, x_{2n} + \frac{h}{2}\dot{x}_{2n} + \frac{h}{4}m_{o2},$$

$$x_{3n} + \frac{h}{2}\dot{x}_{3n} + \frac{h}{4}m_{o3}) \quad (142)$$

$$m_{3i} = hf_i(t_n + h, x_{1n} + h\dot{x}_{1n} + \frac{h}{2}m_{11}, x_{2n} + h\dot{x}_{2n} + \frac{h}{2}m_{12},$$

$$x_{3n} + h\dot{x}_{3n} + \frac{h}{2}m_{13}) \quad (143)$$

$$x_{i(n+1)} = x_{in} + h\dot{x}_{in} + \frac{h}{6}(m_{oi} + m_{li} + m_{2i}) + O(h^5) \quad (144)$$

$$\dot{x}_{i(n+1)} = \dot{x}_{in} + \frac{1}{6}[m_{oi} + 2(m_{li} + m_{2i}) + m_{3i}] + O(h^5) \quad (145)$$

With the substitution of  $x, y, z$  for  $x_1, x_2, x_3$ , equation (140) and the parameter for equation (141) are computed in Section 7.8; equation (141) and the parameter for equation (142) are computed in Section 7.26; equation (142) and the parameters for equation (143) are computed in Section 7.27; and equation (143) through (145) are computed in Section 7.28.

After integration over two intervals of equal size, the results for the velocity components are compared with an integration over the same intervals using Simpson's rule which is also of fourth order accuracy. Simpson's rule is given on page 73 of reference 5 as:

$$\int_{x_0}^{x_2} f(x) dx = \frac{h}{3} (f_0 + 4f_1 + f_2) - \frac{h^5 f^{IV}(\xi)}{90} \quad (146)$$

$$\text{where } x_0 < \xi < x_2$$

or in the notation of this problem:

$$\left. \begin{aligned} \dot{x}_{n+2} &= \dot{x}_n + \frac{\Delta t}{3} [\ddot{x}_n + 4\ddot{x}_{n+1} + \ddot{x}_{n+2}] \\ \dot{y}_{n+2} &= \dot{y}_n + \frac{\Delta t}{3} [\ddot{y}_n + 4\ddot{y}_{n+1} + \ddot{y}_{n+2}] \\ \dot{z}_{n+2} &= \dot{z}_n + \frac{\Delta t}{3} [\ddot{z}_n + 4\ddot{z}_{n+1} + \ddot{z}_{n+2}] \end{aligned} \right\} \quad (147)$$

where the subscript  $n$  means evaluated at time  $t_n$ , the subscript  $n+1$  means evaluated at  $t_n + \Delta t$ , and the subscript  $n+2$  means evaluated at  $t_n + 2\Delta t$ .

By virtue of the comparison between the two integrated results, decisions, are made by the program concerning the accuracy of the integration, and the computing interval for the next two intervals is chosen. The logic underlying these program decisions will now be explained using one first order differential equation as an example.

Let the differential equation to be solved be of the form:

$$\dot{x} = \dot{x}(t, x) \quad (148)$$

If this equation is integrated over an interval,  $h$ , by Runge-Kutta methods of fourth order, then the numerical value of that function corresponds to a Taylor series expansion with an error term of  $O(h^5)$ , i.e.:

$$x_{n+1} = x_n + \dot{x}_n h + \frac{\ddot{x}_n h^2}{2!} + \frac{\dddot{x}_n h^3}{3!} + \frac{x^{IV} h^4}{4!} + O(h^5) \quad (149)$$

The complete functional form of the coefficient of the error term is unknown but it is known to contain  $x^V$ . For the purposes of this program the coefficient of the fifth order term is assumed to be the next term in the Taylor series  $\frac{x^V}{5!}$  and  $x^V$  is assumed to be a slowly varying function. The coefficient of the fifth order term in Simpson's rule is known to be  $-\frac{x^V}{90}$ . Thus if we let  $x_c$

be the correct value of  $x$  at the end of the two equal intervals, and let  $x_{RK}$  and  $x_{SR}$  be the Runge-Kutta and Simpson's rule integrated values respectively, we may write:

$$x_c = x_{RK} + 2\left(\frac{x^V h^5}{5!}\right) \quad (150)$$

$$x_c = x_{SR} - \frac{x^V h^5}{90} \quad (151)$$

Eliminating  $x_c$  between these two equations and solving for  $x^V$  results in:

$$x^V = \frac{36(x_{SR} - x_{RK})}{5h} \quad (152)$$

From equations (150) through (152) the error in the Runge-Kutta solution is estimated to be:

$$\delta x = \frac{3}{5} (x_{SR} - x_{RK}) \quad (153)$$

A factor of  $\frac{3}{5}$  is dropped in the use of this equation in Section 7.14 because an arbitrary constant is introduced at this point.

Letting  $\Delta\dot{x}$ ,  $\Delta\dot{y}$ ,  $\Delta\dot{z}$  be the changes in the  $\dot{x}$ ,  $\dot{y}$ ,  $\dot{z}$  values over the double interval, then what is required in the program is that:

$$E = \text{maximum } (|\delta\dot{x}|, |\delta\dot{y}|, |\delta\dot{z}|) < E_{all} = \text{maximum } (|W_8|C_{max}), \\ 10^{-9} \text{ maximum } (|\dot{x}|, |\dot{y}|, |\dot{z}|) \quad (154)$$

$$\text{where } C_{max} = \text{maximum } (|\Delta\dot{x}|, |\Delta\dot{y}|, |\Delta\dot{z}|) \quad (155)$$

and  $W_8$  is an input number designed to require a series truncation greater than number truncation but as small as possible. An error which is less than

$10^{-9}$  of the maximum of the absolute values of  $\dot{x}$ ,  $\dot{y}$ , and  $\dot{z}$  is always acceptable, since it will be lost in the first addition anyway because of the limits of machine word length.

If  $E \leq E_{\text{all}}$ , the computation proceeds. If  $E > E_{\text{all}}$ , the last two steps are done over.

If  $E$  is greater than an input minimum error  $E_{\text{min}} \cdot C_{\text{max}}$ , then  $\Delta t$  is computed by:

$$\Delta t_{\text{new}} = \text{FDT} * \Delta t_{\text{old}} \left( \frac{E_{\text{all}}}{E} \right)^{.25} \quad (156)$$

If it assumed that  $E_{\text{all}} = W_8 C_{\text{max}} = K \Delta t$ , where  $K$  is some constant (since  $x$  is roughly proportional to  $\Delta t$  and  $C_{\text{max}}$  is normally proportional to  $\Delta x$ ) and  $\text{FDT} = 1$ , then by equations (150), (153), and (154)  $\Delta t_{\text{new}}$  would result in an error of exactly  $E_{\text{all}}$ .  $\text{FDT}$  is an input number  $< 1$  to prevent  $\Delta t_{\text{new}}$  from resulting in an error  $E > E_{\text{all}}$  due to number truncation or change of  $x^V$  over the two new intervals as compared to the  $x^V$  of the previous two intervals.

If  $E < E_{\text{min}} \cdot C_{\text{max}}$ , then it is assumed that the error in  $x_{\text{RK}}$  is primarily due to number truncation in the computations. In this case equation (150) does not apply. The new computing interval is then computed by:

$$\Delta t_{\text{new}} = \Delta t_{\text{min}} \cdot \Delta t_{\text{old}} \quad (157)$$

where  $\Delta t_{\text{min}}$  is an input quantity  $> 1$ . The checks against  $E_{\text{all}}$  are made in Sections 7.14 and 7.15 and checks against  $E_{\text{min}}$  are made in Section 7.22. The computation of the time interval by formula (156) is accomplished in Section 7.19. the computation of  $\Delta t$  by equation (156) is done in Section 7.24.

Section 4  
EQUATIONS IN ORDER OF SOLUTION

4.1 PRINT EXPLANATION OF ERROR CODE

| <u>Error Code</u> | <u>Reason for Halt of Run</u>                 |
|-------------------|---|
| 1                 | Semi-major axis = 0                           |
| 2                 | Eccentricity equals or exceeds 1              |
| 7                 | S (Inverse of radius vector) = 0              |
| 8                 | Derivative of time with respect to $\psi$ = 0 |

4.1.1 Set Input Arrays = 0

Set  $W_I = 0$  for  $I = 1, 2, \dots, 30$

4.2 PROGRAM INPUT

Up to 30 quantities can be input to this program. Those specified thus far are given below.

4.2.1 Osculating Orbital Elements

$W_1 = a$ , semi-major axis in units of  $\mu^{1/2} D$

$W_2 = e$ , eccentricity

$W_3 = i$ , inclination (degrees)

4.2.2 Planetary Mass Ratio

$W_4 = \mu \equiv \epsilon^4$

4.2.3 Time and Accuracy Specifications

$W_5 = FTD$ , modifier of estimated computing interval for the next pass

$W_6 =$  Not used

$W_7$  = Run stop time

$W_8$  = Maximum allowable velocity error

$W_9$  = Modifier of previous computing interval if velocity error is less than the minimum allowable

$W_{10}$  = Initial value of computing interval

$W_{11}$  = Maximum number of failures permitted in computing interval determination

$W_{12}$  = Minimum allowable velocity error

$W_{13}$  = Flag to call for print

If  $W_{13} \leq 0$ , print will be suppressed.

If  $W_{13} > 0$ , print will be called for.

#### 4.3 MOVE INPUT TO WORKING STORAGE AND CHECK INPUT

##### 4.3.1 Move Input to Working Storage and Form Necessary Constants

$$a = |\text{input } a|$$

$$a_2 = a^2$$

$$a_s = a$$

$$e = |\text{input } e|$$

$$e_s = e$$

$$i^o = \text{input } i$$

$$\epsilon_4 = \text{input } \mu$$

$$T_f = W_7$$

#### 4.3.2 Check on Constant Divisors, Halt Run if Any Are Zero

##### 4.3.2.1 Test Semi-Major Axis

If  $a \leq 0$ , go to Step 4.3.2.1.1.

If  $a > 0$ , go to Step 4.3.2.2.

###### 4.3.2.1.1 Set Error Code = 1

###### 4.3.2.1.2 Print Error Code and go to Step 4.2

##### 4.3.2.2 Compute and Test $e_2$

$$e_2 = 1 - e^2$$

If  $e_2 \leq 0$ , go to Step 4.3.2.2.1.

If  $e_2 > 0$ , go to Step 4.4.

###### 4.3.2.2.1 Set Error Code to 2 and go to Step 4.3.2.1.2

#### 4.4 COMPUTE CONSTANTS

$$a_e = ae_2$$

$$a_e^{1/2} = \sqrt{a_e}$$

$$e_2 = \sqrt{e_2}$$

$$a_3 = a^3$$

$$\epsilon_2 = \sqrt{\epsilon_4}$$

$$T_{num\ 1} = 3$$

$$T_{den\ 1} = 1$$

$$\left. \begin{array}{l} T_{num\ i} = T_{num\ i-1} + 2 \\ T_{den\ i} = T_{den\ i-1} + 1 \end{array} \right\} \begin{array}{l} \text{Form for} \\ i = 2, 3, \dots, 40 \end{array}$$

$$ITP = 2$$

$$P_{tl} = 0$$

$$ap_i = 0 \quad i = 1, \dots, 13$$

$$a_{32} = \sqrt{a^3}$$

$$a_{92} = (a^{3/2})^3$$

$$a_{S32} = a_{32}$$

$$\epsilon = \sqrt{\epsilon_2}$$

$$c_1 = 3.75 \epsilon a_3 e$$

$$v_1 = \frac{-a_{32} (11 + 24e)}{12} \quad (35)$$

$$\epsilon_a = \epsilon a_{32}$$

$$\lambda_1 = - .75 \cdot a_{32} \quad (36)$$

$$m_1 = \frac{-a_{32} (1 + 12e)}{6} \quad (34), (35), (36)$$

$$\omega_1 = \frac{a_3 (-7 + 24e)}{12} \quad (46)$$

$$p_3 = \frac{1}{1 + \epsilon_2 \omega_1} - \epsilon_a [2 + \epsilon(v_1 - 2\lambda_1)] \quad (38)$$

$$i = i^{\circ}/57.2957795$$

$$c_i = \cos i$$

$$s_i = \sin i$$

$$i_s = i$$

$$c_{is} = \cos i_s$$

$$s_{is} = \sin i_s$$

$$\dot{\omega} = (\frac{3}{2} \epsilon_2 a_{32} - \epsilon_4 v_1 \omega_1) / (1 + \epsilon_2 \omega_1) \quad (88)$$

$$\dot{\Omega} = - \frac{3}{4} \epsilon + \epsilon_a \quad (87), (36)$$

$$c_{\omega\Omega} = \dot{\omega}\dot{\Omega}$$

$$c_{\omega^2\dot{\Omega}^2} = \dot{\omega}^2 + \dot{\Omega}^2$$

$$a_{12} = a^{1/2}$$

$$p_1 = \frac{1}{1 + \epsilon_2 \omega_1} - \epsilon_a (1 + \epsilon_1 m_1) \quad (37)$$

$$c_2 = 1.875 \epsilon_1 a_{12} e$$

$$c_3 = \frac{.375 \epsilon_1 i a a_{32}}{1 - \epsilon_1 a_{32}}$$

$$c_6 = \dot{\Omega}$$

$$c_7 = \dot{\Omega}^2$$

$$c_8 = 1 - \epsilon_4$$

$$c_9 = a_e a_{12} e_2$$

$$c_{10} = \frac{\epsilon_2 \omega_1}{1 + \epsilon_2 \omega_1} + \epsilon_1 a_{32} (1 + \epsilon_1 m_1)$$

$$c_{11} = 2p_1 - 1$$

$$c_{12} = \frac{2\dot{\Omega}}{a_e^{1/2}}$$

$$c_{13} = \frac{2\dot{\omega}}{a_e^{1/2}}$$

$$c_{14} = 2c_{\omega\dot{\Omega}}$$

Compute  $- \frac{2}{a_e^{1/2}} (\dot{\Omega} + \dot{\omega} \cos i)$

$$c_{15} = -c_{12} - c_{13} c_i$$

Compute  $-2\ddot{\omega\Omega} - (\dot{\omega}^2 + \dot{\Omega}^2) \cos i$

$$c_{16} = -c_{14} - c_{\omega^2 \dot{\Omega}^2} \cdot c_i$$

Compute  $\frac{2}{a_e^{1/2}} (\dot{\Omega} \cos i + \dot{\omega})$

$$c_{17} = c_{12} c_i + c_{13}$$

$$\text{Compute } 2\dot{\omega}\dot{\Omega} \cos i + \dot{\omega}^2 + \dot{\Omega}^2$$

$$c_{18} = c_{14} c_i + c_{\omega^2 \dot{\Omega}^2}$$

$$\text{Compute } \dot{\omega}^2 \sin i$$

$$c_{19} = \dot{\omega}^2 s_i$$

$$\text{Compute } \frac{2e \sin i}{a_e^{1/2}} \dot{\omega}$$

$$c_{20} = e s_i c_{13}$$

$$\text{Compute } \frac{2 \sin i}{a_e^{1/2}} \dot{\omega}$$

$$c_{21} = s_i c_{13}$$

$$\text{Compute } - \frac{2e}{a_e^{1/2}} (\dot{\Omega} + \dot{\omega} \cos i)$$

$$c_{22} = e c_{15}$$

$$\text{Compute } \frac{2e}{a_e^{1/2}} (\dot{\Omega} \cos i + \dot{\omega})$$

$$c_{23} = e c_{17}$$

$$c_{24} = \sqrt{1 - e_s^2}$$

$$T_1 = 2\pi a_{32} + c_1 \sin 2\pi c_{11} \quad (28a)$$

$$- 1.375 e_2 a_{92} \sin 4\pi p_1$$

$$P_{ss} = 2\pi a_{32}$$

$$P_\Omega = 2\pi/\dot{\Omega} \quad (39b)$$

$$P_\omega = 2\pi/\dot{\omega} \quad (72a)$$

$$\Delta E_{max} = 0$$

$$\Delta E_{pmax} = 0$$

$$\alpha_1 = 0$$

$$\theta_1 = 0$$

$$\phi_1 = 0$$

$$C_\alpha = 1$$

$$S_\alpha = 0$$

$$C_\theta = 1$$

$$S_\theta = 0$$

#### 4.5 PRINT CONSTANTS

|          |          |          |          |          |
|----------|----------|----------|----------|----------|
| $w_1$    | $w_2$    | $w_3$    | $w_4$    | $w_5$    |
| $w_6$    | $w_7$    | $w_8$    | $w_9$    | $w_{10}$ |
| $w_{11}$ | $w_{12}$ | $w_{13}$ | $w_{14}$ | $w_{15}$ |

$$\begin{array}{cccccc}
 w_1 & v_1 & \lambda_1 & p_1 & p_3 \\
 c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
 c_7 & c_8 & c_9 & c_{10} & c_{11} & \sqrt{1 - e^2} \\
 \dot{w} & \dot{v} & \ddot{w} & \dot{\Omega}^2 + \dot{\omega}^2
 \end{array}$$

4.6 PRINT HEADER FOR NUMERICAL RESIDUAL PERTURBATION SOLUTION LISTING THE INPUT VALUES OF  $a$ ,  $e$ ,  $i$ , and  $\mu$

4.7 SET PERTURBATION-TOTAL FLAG

$ICH = 1$ ; the signal that residual perturbation equations are to be numerically solved and that the analytic solution is to be computed.

4.8 COMPUTE INITIAL VALUES OF POSITION, VELOCITY, AND ACCELERATION

$$\text{Set } e_3 = 1 + e$$

$$\frac{dt_o}{d\psi} = \frac{a_{32}(e_3)^{1.5}}{(1 - e)^{1/2}} \quad (59)$$

$$\epsilon^2 \frac{dt_1}{d\psi} = \frac{7}{12} \epsilon_2 a_{92}$$

$$\frac{dt}{d\psi} = \frac{dt_o}{d\psi} + \epsilon_2 \frac{dt_1}{d\psi} \quad (59)$$

4.8.1 Test Value of  $\frac{dt}{d\psi}$

If  $\frac{dt}{d\psi} = 0$ , go to Step 4.17.8.1.

If  $\frac{dt}{d\psi} \neq 0$ , continue below.

4.8.2 Compute  $\frac{d\phi}{dt}$  and Set Initial Values of  $t$  and  $\psi$

$$\frac{d\psi}{dt} = 1/\frac{dt}{d\psi}$$

$$\frac{d\phi}{dt} = \frac{\frac{d\psi}{dt}}{(1 + \epsilon_2 \omega_1)} - \epsilon_2 v_1 \quad (65)$$

$$t = 0$$

$$\psi = 0$$

4.8.3 Position Coordinates

$$x_a'' = ae_3, \text{ where } e_3 = 1 + e \quad (61)$$

$$y_a'' = z_a'' = 0 \quad (62)$$

$$x_a' = x_a''$$

$$y_a' = z_a' = 0 \quad (63)$$

$$x' = x_a'$$

$$y' = z' = 0 \quad (109)$$

$$\Delta x' = \Delta y' = \Delta z' = 0 \quad (118)$$

$$\rho = x'$$

$$r = 1 + \epsilon_2 x' \quad (t = 0) \quad (8)$$

$$r^3 = (1 + \epsilon_2 x')^3$$

$$x'_s = x'_a$$

$$y'_s = z'_s = 0 \quad (113)$$

#### 4.8.4 Velocity Components

$$\dot{x}''_a = 0 \quad (64)$$

$$\dot{y}''_a = x'_a \frac{d\phi}{dt}$$

$$\dot{z}''_a = 0$$

$$\dot{y}'_a = \dot{y}''_a c_i + x''_a \dot{\Omega} \quad (66)$$

$$\dot{z}'_a = \dot{y}''_a s_i$$

$$\dot{x}' = 0$$

$$\dot{y}' = \dot{y}'_a \quad (117)$$

$$\dot{z}' = \dot{z}'_a$$

$$\Delta \dot{x} = 0$$

$$\dot{G} = \frac{- \left( \frac{dt_o}{d\psi} \epsilon_2 \omega_1 + \epsilon_2 \frac{dt_1}{d\psi} (1 + \epsilon_2 \omega_1) \right)}{\frac{dt}{d\psi} \frac{dt_o}{d\psi} (1 + \epsilon_2 \omega_1)} - \epsilon_2 \omega_1 - \dot{\omega}$$

$$\Delta \dot{y}' = x' \dot{G} C_i$$

$$\Delta \dot{z}' = x' \dot{G} S_i \quad (119)$$

$$\dot{\rho} = (\dot{y}'^2 + \dot{z}'^2)^{1/2}$$

$$\dot{x}'_s = 0 \quad (114)$$

$$\dot{y}'_s = x' [\dot{\Omega} + \frac{1}{\frac{dt_o}{d\psi}} + \dot{\omega}] C_i \quad (115)$$

$$\dot{z}'_s = x' \frac{1}{\frac{dt_o}{d\psi}} + \dot{\omega} S_i \quad (116)$$

#### 4.8.5 Acceleration Components

$$E_{2x} = \epsilon_2 x$$

$$\begin{aligned} \ddot{\Delta x}' &= c_8 E_{2x} [2 + E_{2x} (3 + E_{2x})]/r^3 \\ &+ 2(\dot{\omega} + \dot{\Omega} C_i) \sqrt{\frac{(1-e)}{a(1+e)}} \\ &+ x'_s (2 C_{\omega\dot{\Omega}} C_i + C_{\omega^2 \dot{\Omega}^2}) \end{aligned} \quad (121)$$

$$\ddot{\Delta y}' = \ddot{\Delta z}' = 0 \quad (122)$$

#### 4.8.6 Initial Value of Jacobi Constant

$$C_{\text{part}} = x'^2 \{(\frac{d\phi}{dt})^2 + \dot{\Omega}^2 - 2[\epsilon \dot{\Omega} - \frac{d\phi}{dt} (\dot{\Omega} - \epsilon) \cos i]\} - 2[\frac{1}{x'} + c_8 x'] \quad (68b)$$

$$c_{\text{init}} = c_{\text{part}} - \frac{2 c_8}{\epsilon_2 r} \quad (68a)$$

$$x'_o = x'$$

$$r_o = 1 + \epsilon_2 x'$$

$$(\Delta c/c)_{\text{approx}} = (\Delta c/c)_{\text{exact}} = 0$$

Print Jacobi integral,  $c_{\text{init}}$ .

#### 4.9 SET RUN START FLAGS

IP = 1

IPRINT = 2

KHALT = 1

KR = 1

##### 4.9.1 Test Perturbation-Total

If ICH = 1, go to Step 4.10.

If ICH = 2, go to Step 4.19.4.

#### 4.10 OUTPUT OF NUMERICAL RESIDUAL PERTURBATION SOLUTION

The format will be:

|                                      |             |                   |                    |              |
|--------------------------------------|-------------|-------------------|--------------------|--------------|
| t                                    | $\Delta x'$ | $\Delta \dot{x}'$ | $\Delta \ddot{x}'$ | $\rho$       |
|                                      | $\Delta y'$ | $\Delta \dot{y}'$ | $\Delta \ddot{y}'$ | $\dot{\rho}$ |
| $\frac{\Delta C_a}{C_{\text{init}}}$ | $\Delta z'$ | $\Delta \dot{z}'$ | $\Delta \ddot{z}'$ |              |

$$\begin{array}{cccc}
 x' & \dot{x}' & x'_a & \dot{x}'_a \\
 y' & \dot{y}' & y'_a & \dot{y}'_a \\
 \frac{\Delta C_E}{C_{init}} & z' & \dot{z}' & z'_a & \dot{z}'_a
 \end{array}$$

If this is the initial point (IP = 1), then go to Step 4.12; otherwise, go to Step 4.18.1.

#### 4.11 TEST HALT FLAG, KHALT

If KHALT = 1, continue numerical residual perturbation at Step 4.12.

If KHALT = 2, go to Section 4.14.1 to discontinue the solution.

If KHALT = 3, initiate numerical solution of total equations of motion at Step 4.19.

#### 4.12 CALL NUMERICAL SOLUTION SUBROUTINE FOR RESIDUAL PERTURBATION EQUATIONS

The arguments for subroutine RKSIMP are:

|  |  |
|--|--|
| KR   | Runge-Kutta flag                               |
| IP   | Initial point flag                             |
| KC   | Simpson's rule flag                            |
| t  | Time   |
| $\Delta t$   | Computing interval                             |
| ICH  | Perturbation-total flag                        |
| $\Delta x'$ , $\Delta y'$ , $\Delta z'$<br>$\dot{\Delta x}'$ , $\dot{\Delta y}'$ , $\dot{\Delta z}'$<br>$\ddot{\Delta x}'$ , $\ddot{\Delta y}'$ , $\ddot{\Delta z}'$ | Position, velocity,<br>acceleration components |

|       |               |
|-------|---------------|
| KHALT | Halt flag     |
| $t_p$ | Print time    |
| $T_f$ | Run stop time |

#### 4.13 TEST RUNGE-KUTTA FLAG, KR

If KR = 1, go to Step 4.14.  
 If KR = 2 or 4, go to Step 4.15.  
 If KR = 3, go to Step 4.16.  
 If KR = 5, go to Step 4.17.

#### 4.14 TEST HALT FLAG, KHALT

If KHALT = 1 or 3, go to Step 4.16.  
 If KHALT = 2, go to Step 4.14.1.

##### 4.14.1 Print Computing Interval Selection Failed

Set  $T_f$  = last print time.

If ICH = 1, initiate numerical solution of total equations of motion  
 at Step 4.19.  
 If ICH = 2, begin plotting results (Step 4.31).

#### 4.15 COMPUTE TWO-BODY EQUATIONS

##### 4.15.1 Solve Kepler's Equations

###### 4.15.1.1 Form Constant

$$t_{a32} = t_s / a_{32}$$

4.15.1.2 Compute a First Estimate for Eccentric Anomaly

$$E = t_{a32} - e_s \sin(t_{a32})[1 + e_s \cos(t_{a32})] \quad (108)$$

$$\Delta E = 10,000$$

4.15.1.3 Compute New Guess at E

$$S_E = \sin E$$

$$C_E = \cos E$$

$$D_n = 1 + e_s C_E$$

$$E_n = \frac{t_{a32} + e_s (EC_E - S_E)}{D_n}$$

If  $E_n = E$ , go to Step 4.15.2.

If  $|E_n - E| \geq \Delta E$ , go to Step 4.15.1.4; otherwise set  $E = E_n$ ,  
 $\Delta E = |E_n - E|$  and repeat Step 4.15.1.3.

4.15.1.4 Store Maximum  $\Delta E$

$$\Delta E_{\max} = \max(|\Delta E|, \Delta E_{\max})$$

4.15.2 Compute Sine and Cosine of the True Anomaly

$$\sin v = \sqrt{\frac{1 - e_s^2}{1 + e_s \cos E}} \sin E = \frac{C_{24} S_E}{D_n} \quad (105)$$

$$\cos v = \frac{\cos E + e_s}{1 + e_s \cos E} = \frac{C_E + e_s}{D_n} \quad (104)$$

4.15.3 Compute  $\rho_s = a_s(1 + e C_E)$  (106)

4.15.4 Test Perturbation-Total Flag

If ICH = 1, continue below.

If ICH = 2, go to Step 4.28.2.

4.15.5 Compute Arguments of Node and Perigee and Their Sines and Cosines

$$\Omega = \dot{\Omega} t_\Omega; \quad C_\Omega = \cos \Omega; \quad S_\Omega = \sin \Omega$$

$$\omega = \dot{\omega} t_\omega; \quad C_\omega = \cos \omega; \quad S_\omega = \sin \omega$$

$$C_{\phi_s} = \cos v \cos \omega - \sin v \sin \omega$$

$$S_{\phi_s} = \sin v \cos \omega + \cos v \sin \omega$$

4.15.6 Compute the Precessing Two-Body Position Coordinates in the Inertial Coordinate System

$$x'_s = \rho_s (C_{\phi_s} C_\Omega - S_{\phi_s} S_\Omega C_i)$$

$$y'_s = \rho_s (C_{\phi_s} S_\Omega + S_{\phi_s} C_\Omega C_i)$$

$$z'_s = \rho_s S_{\phi_s} S_i$$

$$C_{et} = \cos \gamma$$

$$S_{et} = \sin \gamma$$

4.16 ACCELERATION COMPONENTS FOR NUMERICAL RESIDUAL PERTURBATION EQUATIONS OF MOTION

4.16.1 Computation of the Small Differences Between the Two-Body Terms

$$\text{Set } q = \frac{(x'_s + \frac{\Delta x'}{2})\Delta x' + (y'_s + \frac{\Delta y'}{2})\Delta y' + (z'_s + \frac{\Delta z'}{2})\Delta z'}{p_s^2} \quad (101)$$

Determine the term of lowest order in the following series which is less than  $2 \times 10^{-9}$

$$- \frac{5}{2!} q + \frac{5 \cdot 7}{3!} q^2 - \frac{5 \cdot 7 \cdot 9}{4!} q^3 + \dots$$

up to a limit of 40 terms. If the fortieth term still exceeds  $2 \times 10^{-9}$ , print its value, set final time  $t_f = p_t(I_{TP} - 1)$ , and begin the solution of the total equations of motion at Step 4.19. If the absolute value of the nth term is  $< 2 \times 10^{-9}$  and  $n \leq 40$ , then compute

$$fq = 3q \left\{ 1 - \frac{5}{2} q \left( 1 - \frac{7}{3} q \left( 1 - \frac{9}{4} q \left( 1 - \frac{11}{5} q \dots \right) \right) \right) \right\} \quad (102)$$

etc., to the nth term

4.16.2 Compute Numerical Residual Perturbation Acceleration Components

$$x' = x'_s + \Delta x'$$

$$y' = y'_s + \Delta y' \quad (96)$$

$$z' = z'_s + \Delta z'$$

$$p^2 = x'^2 + y'^2 + z'^2$$

$$r^2 = 1 + \epsilon_2 [2(x' c_{et} + y' s_{et}) + \epsilon_2 p^2] \quad (8)$$

$$r = \sqrt{r^2}$$

$$r^3 = r^2 r$$

$$E_{r_1} = C_8 \epsilon_2 / r^3$$

$$E_{r_2} = C_8 \epsilon_2 [2(x' c_{\epsilon t} + y' s_{\epsilon t}) + \epsilon_2 \rho^2] [1 + r(1 + r)] / r^3 (1 + r)$$

$$\rho_s^3 = (\rho_s)^3$$

$$\begin{aligned} \text{Compute } B_1 &= - \frac{2(\dot{\Omega} + \dot{\omega} \cos i)}{\sqrt{a(1 - e^2)}} + \rho_s [2\dot{\omega}\dot{\Omega} + (\dot{\omega}^2 + \dot{\Omega}^2) \cos i] \\ &= C_{15} + \rho_s C_{16} \end{aligned}$$

$$\begin{aligned} \text{Compute } B_2 &= \frac{2(\dot{\Omega} \cos i + \dot{\omega})}{\sqrt{a(1 - e^2)}} + \rho_s [2\dot{\omega}\dot{\Omega} \cos i + (\dot{\omega}^2 + \dot{\Omega}^2)] \\ &= C_{17} + \rho_s C_{18} \end{aligned}$$

Following the preliminary computations, the acceleration components may be computed as:

$$\begin{aligned} \ddot{x}' &= (f_q x' - \Delta x') / \rho_s^3 - E_{r_1} x' + E_{r_2} c_{\epsilon t} \\ &\quad + S_{\phi_s} S_{\Omega} B_1 + C_{\phi_s} C_{\Omega} B_2 - (S_{\omega} S_{\Omega} C_{22} + C_{\omega} C_{\Omega} C_{23}) \quad (98) \end{aligned}$$

$$\begin{aligned} \ddot{y}' &= (f_q y' - \Delta y') / \rho_s^3 - E_{r_1} y' + E_{r_2} s_{\epsilon t} \\ &\quad - S_{\phi_s} C_{\Omega} B_1 + C_{\phi_s} S_{\Omega} B_2 + (S_{\omega} C_{\Omega} C_{22} + C_{\omega} S_{\Omega} C_{23}) \quad (99) \end{aligned}$$

$$\begin{aligned}\ddot{\Delta z'} &= (\dot{f}_q z' - \Delta z')/\rho_s^3 - E_{r_1} z' + S_{\phi_s} S_i (C_{13} + \rho_s \omega^2) - e s_i S_\omega C_{13} \\ &= (\dot{f}_q z' - \Delta z')/\rho_s^3 - E_{r_1} z' + S_{\phi_s} (C_{21} + \rho_s C_{19}) - S_\omega C_{20}\end{aligned}\quad (100)$$

where  $C_\Omega$ ,  $S_\Omega$ ,  $C_\omega$ ,  $S_\omega$ ,  $C_{\phi_s}$ , and  $S_{\phi_s}$  are defined in Section 4.15.5,  $C_{et}$  and  $S_{et}$  in Section 4.15.6, and  $C_{13}$  and  $C_{19}$  through  $C_{23}$  in Section 4.4.

#### 4.16.3 Test Runge-Kutta Flag, KR

If  $KR = 1$ , go to Step 4.16.4.

If  $KR > 1$ , go to Step 4.12.

#### 4.16.4 Test Print Flag, IPRINT

If  $IPRINT = 2$ , go to Step 4.12.

If  $IPRINT \neq 2$ , go to Step 4.17.

### 4.17 COMPUTATIONS FOR PRINT ONLY

#### 4.17.1 Reduce All Angles that Increase with Time by Integer Multiples of $2\pi$

$$T_s = T_s, \text{ modulo } P_{ss} \quad (108)$$

$$T_\Omega = T_\Omega, \text{ modulo } P_\Omega \quad (39a, b)$$

$$T_\omega = T_\omega, \text{ modulo } P_\omega \quad (72a)$$

$$\gamma = \gamma, \text{ modulo } 2\pi$$

4.17.1.1 Test Value of  $T_\psi$

If  $t_\psi \geq T_1$ , go to Step 4.17.1.2.

If  $t_\psi > T_1$ , go to Step 4.17.2.1.

(28b)

4.17.1.2 Set  $T_\psi = T_\psi - T_1$

Recompute  $T_1$

$$\phi_1 = [\phi_1 + \frac{2\pi}{1 + \epsilon_2 \omega_1} - \epsilon_2 v_1 T_1] \text{ modulo } 2\pi \quad (49)$$

$$\alpha_1 = \{ [(\alpha_1 + 2\pi p_1) \text{ modulo } 2\pi] + 2\pi p_1 \} \text{ modulo } 2\pi \quad (28d)$$

$$S_{\alpha_1} = \sin \alpha_1$$

$$C_{\alpha_1} = \cos \alpha_1$$

$$\alpha_2 = \{ [(\alpha_1 + 2\pi p_1) \text{ modulo } 2\pi] + 2\pi p_1 \} \text{ modulo } 2\pi \quad (28g)$$

$$\theta_1 = (\theta_1 + 2\pi C_{11}) \text{ modulo } 2\pi \quad (28c)$$

where  $C_{11}$  is defined in Section 4.4.

$$S_{\theta_1} = \sin \theta_1$$

$$C_{\theta_1} = \cos \theta_1$$

$$\theta_2 = (\theta_1 + 2\pi C_{11}) \text{ modulo } 2\pi \quad (28f)$$

$$T_1 = 2\pi a_{32} + C_1 (\sin \theta_2 - S_{\theta_1}) - 1.375 e_2 a_{92} (\sin \alpha_2 - S_{\alpha_1}) \quad (28e)$$

where  $C_1$  is defined in Section 4.4.

$$\beta = (\beta + 2\pi p_3) \text{ modulo } 2\pi \quad (29)$$

#### 4.17.2 Solution of the Perturbed Kepler Equation

As in the unperturbed case,  $\psi$  (true anomaly) is an implicit function of time and  $\psi(t)$  must be found by iteration.

##### 4.17.2.1 Compute Initial Guess at $E$

$$E = t_\psi / a_{32} - e \sin(t_\psi / a_{32})(1 - e \cos t_\psi / a_{32})$$

$$\Delta E_p = 10,000$$

##### 4.17.2.2 Compute Sine and Cosine of $\psi$

$$S_E = \sin E$$

$$C_E = \cos E$$

$$D_n = 1 + e C_E$$

$$S_\psi = \sin \psi = \frac{e_2 S_E}{D_n} \quad (68aa)$$

where  $e_2 = \sqrt{1 - e^2}$  is computed in Section 4.4

$$C_\psi = \cos \psi = \frac{C_E + e}{D_n} \quad (68ab)$$

#### 4.17.2.3 Compute $\psi$

##### 4.17.2.3.1 Test $\sin \psi$

If  $\sin \psi = 0$ , go to Step 4.17.2.3.2; otherwise go to Step 4.17.2.3.3.

##### 4.17.2.3.2 Set $\psi = 0$ or $\pi$

$$\psi = \frac{\pi}{2} (1 - 1 \operatorname{sign} C_\psi)$$

Go to Step 4.17.3.

##### 4.17.2.3.3 Compute principal value of $\psi$ ( $-\pi/2 < \psi < \pi/2$ )

$$\psi = \tan^{-1} \left( \frac{\sin \psi}{\cos \psi} \right)$$

##### 4.17.2.3.4 Test $\cos \psi$

If  $\cos \psi < 0$ , then set  $\psi = \psi + \pi$  and go to Step 4.17.2.4.

If  $\cos \psi = 0$ , go to Step 4.17.2.3.5.

If  $\cos \psi > 0$ , go to Step 4.17.2.3.6.

##### 4.17.2.3.5 Set $\psi = \pi/2$ or $3\pi/2$

$$\psi = \frac{\pi}{2} (2 - 1 \operatorname{sign} S_\psi)$$

Go to Step 4.17.2.4.

##### 4.17.2.3.6 Place $\psi$ in first or third quadrant

$$\psi = \operatorname{sign} E [\psi + \pi(1 - 1 \operatorname{sign} S_\psi + 2 \operatorname{integer part of} \frac{E}{2\pi})] \quad (68ah)$$

#### 4.17.2.4 Compute New Guess at $E$

Compute  $(2p_1 - 1)\psi$

$$C_{11\psi} = C_{11} \cdot \psi$$

Compute  $\sin(2p_1 - 1)\psi$

$$S_{C11\psi} = \sin C_{11\psi}$$

Compute  $\cos(2p_1 - 1)\psi$

$$C_{C11\psi} = \cos C_{11\psi}$$

Compute  $\sin(2p_1 - 1)(\psi + \theta_1)$

$$S_{P1} = S_{C11\psi} C_{\theta_1} + C_{C11\psi} S_{\theta_1}$$

Compute  $\cos(2p_1 - 1)(\psi + \theta_1)$

$$C_{P1} = C_{C11\psi} C_{\theta_1} - S_{C11\psi} S_{\theta_1}$$

Compute  $\cos 2p_1 \psi$

$$C_{2P1\psi} = C_{C11\psi} C_\psi - S_{C11\psi} S_\psi$$

Compute  $\sin 2p_1 \psi$

$$S_{2P1\psi} = S_{C11\psi} C_\psi + C_{C11\psi} S_\psi$$

Compute  $\cos 2p_1(\psi + \alpha_1)$

$$C_{P1\psi} = C_{2P1\psi} C_{\alpha_1} - S_{2P1\psi} S_{\alpha_1}$$

Compute  $\sin 2p_1(\psi + \alpha_1)$

$$S_{P1\psi} = S_{2P1\psi} C_{\alpha_1} + C_{2P1\psi} S_{\alpha_1}$$

$$\epsilon^2 \frac{dt_1}{d\psi} = -c_1(c_\psi - c_{Pl}) + \epsilon_2 a^{9/2} \left[ \frac{10}{3} c_\psi - 2.75 c_{Pl\psi} \right] \quad (68ad)$$

$$\epsilon^2 \frac{dt_1}{dE} = \frac{e_2}{D_n} \epsilon^2 \frac{dt_1}{d\psi} \quad (68ae)$$

$$\epsilon^2 t_1 = -c_1(s_\psi - s_{Pl} + s_{\theta_1}) + \epsilon_2 a_{92} \left[ \frac{10}{3} s_\psi - 1.375(s_{Pl\psi} - s_{\alpha_1}) \right] \quad (68af)$$

$$E_n = \frac{a_{32} e(E \cos E - \sin E) + E \frac{dt_1}{dE} + t_\psi - \epsilon^2 t_1}{a_{32}(1 + e \cos E) + \epsilon^2 \frac{dt_1}{dE}} \quad (68ag)$$

If  $E_n = E$ , go to Step 4.17.3.

If  $|E_n - E| \geq \Delta E_p$ , go to Step 4.17.2.5.

Otherwise set

$$\Delta E_p = |E_n - E|$$

and

$$E = E_n$$

and go to Step 4.17.2.2.

#### 4.17.2.5 Store Maximum Final Value of $\Delta E_p$

$$\Delta E_{Pmax} = \max (\Delta E_{Pmax}, |\Delta E_p|)$$

#### 4.17.3 Compute Auxiliary Functions of $\psi$ and $t$

$$C_{P_3} = \cos p_3 \psi \cos \beta - \sin p_3 \psi \sin \beta \quad (29)$$

$$S_{P_3} = \sin p_3 \psi \cos \beta + \cos p_3 \psi \sin \beta$$

$$s = \frac{1 - e C_\psi}{a_e} + C_2 (C_\psi - C_{Pl}) + \epsilon_2 a_2 [(2 - 5C_\psi)/3 + C_{Pl\psi}] \quad (18, 19, 22)$$

where  $C_2$  is defined in Section 4.4,  $C_\psi$  and  $S_\psi$  in Section 4.17.2.2, and  $C_{Pl}$  and  $C_{Pl\psi}$  in Section 4.17.2.4.

Test the value of  $s$ .

If  $s \leq 0$ , set and print the error code = 7 and go to Step 4.3.2.1.2.

If  $s > 0$ , continue below.

#### 4.17.4 Compute Double Prime Coordinates

$$\phi = \frac{\psi}{1 + \epsilon_2 w_1} - \epsilon_2 v_1 t_\psi \quad (49)$$

$$x''_a = (\cos \phi \cos \phi_1 - \sin \phi \sin \phi_1)/s \quad (48)$$

$$y''_a = (\sin \phi \cos \phi_1 + \cos \phi \sin \phi_1)/s$$

$$z''_a = C_3 (S_{P_3} - p_3 S_\psi) \quad (24)$$

where  $C_3$  is defined in Section 4.4,  $S_{P_3}$  in Section 4.17.3, and  $S_\psi$  in Section 4.17.2.2.

#### 4.17.5 Compute Node

$$\Omega = C_6 T_\Omega \quad \text{where } C_6 = \dot{\Omega} \quad (39)$$

$$C_\Omega = \cos \Omega$$

$$S_{\Omega} = \sin \Omega$$

#### 4.17.6 Compute Approximate Coordinates in Plane of Planets

$$\begin{aligned}x'_a &= x''_a C_{\Omega} - y''_a S_{\Omega} C_i + z''_a S_{\Omega} S_i \\y'_a &= x''_a S_{\Omega} + y''_a C_{\Omega} C_i - z''_a C_{\Omega} S_i \\z'_a &= y''_a S_i + z''_a C_i\end{aligned}\tag{50}$$

where  $S_i = \sin i$  and  $C_i = \cos i$

#### 4.17.7 Compute Time Derivatives of $\psi$ , $\phi$ , and $s$

$$\frac{dt}{d\psi} = C_9 / (1 - eC_{\psi})^2 - C_1(C_{\psi} - C_{Pl}) + \epsilon_2 a_{92} \left( \frac{10}{3} C_{\psi} - 2.75 C_{Pl\psi} \right) \tag{52}$$

where  $C_9$  is defined in Section 4.4 and  $C_{\psi}$ ,  $C_{Pl}$ , and  $C_{Pl\psi}$  are defined in Section 4.17.2.

#### 4.17.8 Test Value of $\frac{dt}{d\psi}$

If  $\frac{dt}{d\psi} = 0$ , go to Step 4.17.8.1.

If  $\frac{dt}{d\psi} \neq 0$ , go to Step 4.17.9.

#### 4.17.8.1 Set the Error Code = 8 and Go to Step 4.3.2.1.2

#### 4.17.9 Compute Time Derivatives of $\phi$ and $s$

$$\frac{d\psi}{dt} = \frac{1}{\frac{dt}{d\psi}} \tag{56}$$

$$\frac{d\phi}{dt} = \frac{d\psi}{dt} (1 + \epsilon_2 \omega_1) - \epsilon_2 v_1 \quad (55)$$

$$\frac{ds}{d\psi} = \frac{eS_\psi}{a_e} - c_2 (s_{P1} - s_\psi) + \epsilon_2 a_2 \left( \frac{5}{3} s_\psi - 2 s_{P1\psi} \right) \quad (51)$$

where  $s_{P1}$ ,  $s_\psi$ , and  $s_{P1\psi}$  are defined in Section 4.17.2 and  $c_2$  is defined in Section 4.4.

$$\frac{ds}{dt} = \frac{ds}{d\psi} \frac{d\psi}{dt}$$

#### 4.17.10 Compute Approximate Velocity Components in Orbit Plane

$$\begin{aligned}\dot{x}_a'' &= y_a'' \frac{d\phi}{dt} - \frac{x_a''}{s} \frac{ds}{dt} \\ \dot{y}_a'' &= x_a'' \frac{d\phi}{dt} - \frac{y_a''}{s} \frac{ds}{dt} \end{aligned} \quad (54)$$

$$\dot{z}_a'' = c_3 (c_{P3} - c_\psi) \frac{d\psi}{dt}$$

where  $c_3$  is defined in Section 4.4,  $c_{P3}$  in Section 4.17.3, and  $c_\psi$  in Section 4.17.2.2.

#### 4.17.11 Compute $\rho$

$$\rho = \sqrt{\rho^2}$$

##### 4.17.11.1 Test Perturbation-Total Flag

If  $ICH = 1$ , go to Step 4.17.12.

If ICH = 2, go to Step 4.17.14.1.

#### 4.17.12 Compute Approximate Velocity Components in Inertial System

$$\dot{x}'_a = C_6 (-x''_a S_\Omega - y''_a C_\Omega C_i + z''_a C_\Omega S_i) + \dot{x}''_a C_\Omega - \dot{y}''_a S_\Omega C_i + \dot{z}''_a S_\Omega S_i$$

$$\dot{y}'_a = C_6 (x''_a C_\Omega - y''_a S_\Omega C_i + z''_a S_\Omega S_i) + \dot{x}''_a S_\Omega + \dot{y}''_a C_\Omega C_i - \dot{z}''_a C_\Omega S_i$$

$$\dot{z}'_a = \dot{y}''_a S_i + \dot{z}''_a C_i \quad (57)$$

where  $C_6 = \dot{\Omega}$ ,  $C_\Omega = \cos \Omega$ ,  $S_\Omega = \sin \Omega$ ,  $C_i = \cos i$ , and  $S_i = \sin i$  as computed previously.

#### 4.17.13 Compute Precessing Keplerian Velocity Components in Inertial System

$$\begin{aligned} \dot{x}'_s &= - [S_{\phi_s} C_\Omega + C_{\phi_s} S_\Omega C_i - e(S_\omega C_\Omega + C_\omega S_\Omega C_i)]/a_e^{1/2} \\ &\quad - \rho_s [\dot{\Omega}(C_{\phi_s} S_\Omega + S_{\phi_s} C_\Omega C_i) + \dot{\omega}(S_{\phi_s} C_\Omega + C_{\phi_s} S_\Omega C_i)] \end{aligned} \quad (91)$$

$$\begin{aligned} \dot{y}'_s &= - [S_{\phi_s} S_\Omega - C_{\phi_s} C_\Omega C_i - e(S_\omega S_\Omega - C_\omega C_\Omega C_i)]/a_e^{1/2} \\ &\quad + \rho_s [\dot{\Omega}(C_{\phi_s} C_\Omega - S_{\phi_s} S_\Omega C_i) - \dot{\omega}(S_{\phi_s} S_\Omega - C_{\phi_s} C_\Omega C_i)] \end{aligned} \quad (92)$$

$$\dot{z}'_s = [(C_{\phi_s} - e C_\omega)/a_e^{1/2} + \rho_s \dot{\omega} C_{\phi_s}] S_i \quad (93)$$

where  $C_\Omega$ ,  $S_\Omega$ ,  $C_\omega$ ,  $S_\omega$ , and  $C_{\phi_s}$  are defined in Section 4.15.5 and  $a_e^{1/2}$ ,

$C_i$ , and  $S_i$  are computed in Section 4.4.

4.17.14 Compute Total Inertial Velocity Components and Vector Magnitudes

$$\dot{x}' = \dot{x}'_s + \Delta \dot{x}'$$

$$\dot{y}' = \dot{y}'_s + \Delta \dot{y}' \quad (117)$$

$$\dot{z}' = \dot{z}'_s + \Delta \dot{z}'$$

4.17.14.1 Compute Inertial Velocity

$$v^2 = \dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2$$

$$\dot{\rho} = \sqrt{v^2}$$

4.17.15 Compute Jacobi Constant Fractional Errors

$$r = \sqrt{r^2}$$

$$\Delta C_E = v^2 + 2[\epsilon_1(\dot{x}'y' - \dot{y}'x') - 1/\rho - c_8(x'C_{\epsilon t} + y'S_{\epsilon t})] \quad (16)$$

$$-c_{\text{part}} + 2c_8 \frac{[2(x'C_{\epsilon t} + y'S_{\epsilon t} - x_o) + \epsilon_2(\rho^2 - x_o^2)]}{rr_o(r_o + r)}$$

where  $v$  is defined in Section 4.17.14.1,  $c_8$  in Section 4.4, and  $C_{\epsilon t}$  and  $S_{\epsilon t}$  in Section 4.15.6.

$$\frac{\Delta C_E}{C} = \Delta C_E / C_{\text{init}}$$

$$\rho_a^2 = x_a'^2 + y_a'^2 + z_a'^2$$

$$r_a = \{1 + \epsilon_2 [2(x'_a c_{\epsilon t} + y'_a s_{\epsilon t}) + \epsilon_2 \rho_a^2]\}^{1/2}$$

$$\Delta C_a = \dot{x}'_a^2 + \dot{y}'_a^2 + \dot{z}'_a^2 + 2[\epsilon_1 (\dot{x}'_a y'_a - \dot{y}'_a x'_a) - \frac{1}{\sqrt{\rho_a^2}}]$$

$$- C_8 (x'_a c_{\epsilon t} + y'_a s_{\epsilon t})] - C_{part} \quad (16)$$

$$+ 2C_8 \frac{[2(x'_a c_{\epsilon t} + y'_a s_{\epsilon t} - x'_o) + \epsilon_2 (\rho_a^2 - x_o^2)]}{r_a r_o (r_o + r_a)}$$

$$\frac{\Delta C_a}{C} = \frac{\Delta C_a}{C_{init}}$$

$$\rho_s^2 = x_s^2 + y_s^2 + z_s^2$$

$$\rho_s = \sqrt{\frac{2}{\rho_s}}$$

$$r_s = \{1 + \epsilon_2 [2(x'_s c_{\epsilon t} + y'_s s_{\epsilon t}) + \epsilon_2 \rho_s^2]\}^{1/2}$$

$$\frac{\Delta C_s}{C} = \{\dot{x}'_s^2 + \dot{y}'_s^2 + \dot{z}'_s^2 + 2[\epsilon_1 (\dot{x}'_s y'_s - \dot{y}'_s x'_s) - 1/\rho_s]$$

$$- C_8 (x'_s c_{\epsilon t} + y'_s s_{\epsilon t})] - C_{part}$$

$$+ 2C_8 \frac{[2(x'_s c_{\epsilon t} + y'_s s_{\epsilon t} - x'_o) + \epsilon_2 (\rho_s^2 - x_o^2)]}{r_s r_o (r_o + r_s)} \} / C_{init}$$

If KHALT = 1, go to Step 4.17.16.

If KHALT = 2, go to Step 4.14.1

If KHALT = 3, go to Step 4.10.

4.17.16 Test if Printing is Desired

If  $W_{13} \leq 0$ , go to Step 4.18.1.

If  $W_{13} > 0$ , go to Step 4.10.

4.18 SAVE PLOT VALUES AND TEST IF PLOTTING ARRAY IS FULL

4.18.1 Save Values from Numerical Solution of Total Equations of Motion

$$P_{t(1TP)} = t$$

$$P_{t(1TP + 1)} = T_f$$

$$IT = 13 \ IT_P - 12$$

$$a_{p(IT)} = \left| \frac{\Delta C_a}{C} \right|$$

$$a_{p(IT + 1)} = \left| \frac{\Delta C_s}{C} \right|$$

$$a_{p(IT + 2)} = x'$$

$$a_{p(IT + 3)} = y'$$

$$a_{p(IT + 4)} = \left| \frac{\Delta C_E}{C} \right|$$

$$a_{p(IT + 5)} = [(x'_{a_1} - x')^2 + (y'_{a_1} - y')^2 + (z'_{a_1} - z')^2]^{1/2} / a$$

$$a_{p(IT + 6)} = [(x'_{s_1} - x')^2 + (y'_{s_1} - y')^2 + (z'_{s_1} - z')^2]^{1/2} / a$$

$$a_{p(IT + 7)} = z'$$

$$a_{p(IT + 8)} = \dot{x}'$$

$$a_p(IT + 9) = a_{12}[(\dot{x}'_a - \dot{x}')^2 + (\dot{y}'_a - \dot{y}')^2 + (\dot{z}'_a - \dot{z}')^2]^{1/2}$$

$$a_p(IT + 10) = a_{12}[(\dot{x}'_s - \dot{x}')^2 + (\dot{y}'_s - \dot{y}')^2 + (\dot{z}'_s - \dot{z}')^2]^{1/2}$$

$$a_p(IT + 11) = \dot{y}'$$

$$a_p(IT + 12) = \dot{z}'$$

Increment index of plot arrays.

#### 4.18.2 Test Value of ITP

If  $ITP > 700$ , set  $T_f = P_{t(ITP)}$ , and go to Step 4.19.

If  $ITP \leq 700$ , set  $ITP = ITP + 1$ , set  $P_{t(ITP)} = T_f$ , and go to Step 4.11.

### 4.19 INITIATE NUMERICAL SOLUTION OF TOTAL EQUATIONS OF MOTION

#### 4.19.1 Set Perturbation-Total Flag

ICH = 2

KT = 0

#### 4.19.2 Compute Initial Conditions for Numerical Solution of Total Equations of Motion

$x' = ae_3$  where  $e_3$  is computed in Section 4.8.

$y' = 0$

$z' = 0$

$\gamma = 0$

$$\rho = x' \quad (59)$$

$$t = 0$$

$$\frac{dt}{d\psi} = \frac{a_{32} e^{1.5}}{\sqrt{1 - e}} + 7 \epsilon_2 a_{92} / 12$$

where  $a_{32}$ ,  $a_{92}$ , and  $\epsilon_2$  are computed in Section 4.4.

$$\frac{d\psi}{dt} = \frac{1}{\frac{dt}{d\psi}}$$

$$\frac{d\phi}{dt} = \frac{d\psi}{dt(1 + \epsilon_2^2 \omega_1)} = \epsilon_2 v_1 \quad (65)$$

$$\dot{x}' = 0$$

$$\dot{y}' = x' \left( \frac{d\phi}{dt} c_i + c_6 \right) \quad (66), (64)$$

$$\dot{z}' = x' \frac{d\phi}{dt} s_i \quad (66), (64)$$

$$\dot{\rho} = (\dot{y}'^2 + \dot{z}'^2)^{1/2}$$

$$r^3 = (1 + \epsilon_2 x')^3$$

$$E_{2x} = \epsilon_2 x'$$

$$\ddot{x}' = - \frac{1}{x'^2} + C_8 E_{2x} (2 + E_{2x} (3 + E_{2x})) / r^3$$

where  $C_8$  is computed in Section 4.4.

$$\ddot{y}' = \ddot{z}' = \frac{\Delta C}{C} = 0 \quad (5)$$

**4.19.3 Compute Semi-Major Axis and Eccentricity of Initial Osculating Ellipse**

$$a_s = \frac{1}{\frac{2}{\rho} - \dot{\rho}^2} \quad (124)$$

$$e_s = \frac{x'}{a_s} - 1 \quad (125)$$

$$a_{s32} = (a_s)^{1.5}$$

$$P_{ss} = 2\pi a_{s32}$$

$$a_{el/2} = \sqrt{a_s(1 - e_s^2)}$$

Go to Step 4.9.

**4.19.4 Print Header for Numerical Solution of Total Equations of Motion**

**4.20 PRINT NUMERICAL SOLUTION OF TOTAL EQUATIONS OF MOTIONS**

t' x'  $\dot{x}'$   $\ddot{x}'$   $\rho$

y'  $\dot{y}'$   $\ddot{y}'$   $\dot{\rho}$

$\frac{\Delta C_c}{C}$  z'  $\dot{z}'$   $\ddot{z}'$

$\frac{\Delta C_a}{C}$   $\frac{\Delta C_s}{C}$   $\frac{\Delta C_c}{C}$   $\frac{\Delta C_F}{C}$   $\frac{\Delta C_E}{C}$

$\Delta \rho_a$   $\Delta \rho_s$   $\Delta \rho_c$   $\Delta \rho_F$

$\dot{\Delta \rho}_a$   $\dot{\Delta \rho}_s$   $\dot{\Delta \rho}_c$   $\dot{\Delta \rho}_F$

4.21 TEST HALT FLAG, KHALT

If KHALT = 1, continue below.

If KHALT = 2, go to Step 4.14.1.

If KHALT = 3, go to Step 4.31.

4.22 CALL NUMERICAL SOLUTION SUBROUTINE FOR TOTAL EQUATIONS OF MOTIONS

The arguments for subroutine RKSIMP are:

KR

IP

KC

t

$\Delta t$

ICH

$x'$ ,  $y'$ ,  $z'$

$\dot{x}'$ ,  $\dot{y}'$ ,  $\dot{z}'$

$x'$ ,  $y'$ ,  $z'$

KHALT

$t_{PT}$

$T_p$

4.23 TEST RUNGE-KUTTA FLAG, KR

If KR = 1, continue below.

If KR = 2, 3, or 4, go to Step 4.25.

If KR = 5, go to Step 4.28.1.

4.24 TEST HALT FLAG, KHALT

If KHALT = 2, set  $t_f = P_{t(ITP - 1)}$  and go to Step 4.14.1.

If KHALT = 1 or 3, continue below.

4.25 COMPUTE TOTAL EQUATIONS OF MOTION

$$\rho^2 = x'^2 + y'^2 + z'^2 \quad (7)$$

$$\rho^3 = (\rho^2)^{1.5}$$

$\gamma = \gamma$  modulo  $2\pi$        $\gamma = \epsilon t$  is developed in subroutine RKSIMP

$$C_{\epsilon t} = \cos \gamma$$

$$S_{\epsilon t} = \sin \gamma$$

$$r^2 = 1 + \epsilon_4 \rho^2 + 2\epsilon_2 (x' C_{\epsilon t} + y' S_{\epsilon t}) \quad (8)$$

$$r = (r^2)^{.5}$$

$$r^3 = r^2 r$$

$$E_{r_1} = C_8 \epsilon_2 / r^3 \quad C_8 \text{ is defined in Section 4.4.}$$

$$E_{r_2} = C_8 \epsilon_2 [2(x' C_{\epsilon t} + y' S_{\epsilon t}) + \epsilon_2 \rho^2] \frac{[1 + r(1 + r)]}{r^3(1 + r)} \quad (5)$$

$$\left. \begin{aligned} \ddot{x}' &= -\frac{x'}{\rho^3} - E_{r_1} x' + E_{r_2} C_{et} \\ \ddot{y}' &= -\frac{y'}{\rho^3} - E_{r_1} y' + E_{r_2} S_{et} \\ \ddot{z}' &= -\frac{z'}{\rho^3} - E_{r_1} z' \end{aligned} \right\} \quad (5)$$

When the expression for  $E_{r_1}$  and  $E_{r_2}$  are substituted in the equations above, one arrives at a set of differential equations identical to (5) except that the last term on the right side of (5) has been put over the common denominator  $r^3(1+r)$  in order to cancel the large terms in the numerator.

#### 4.26 TEST RUNGE-KUTTA FLAG, KR

If KR = 1, continue below.

If KR = 2, 3, 4, or 5, go to Step 4.22.

#### 4.27 TEST PRINT FLAG, IPRINT

If IPRINT = 1, continue below.

If IPRINT = 2 or 3, go to Step 4.22.

#### 4.28 COMPUTE DATA FOR PRINT AND FOR PLOT ROUTINE

##### 4.28.1 Compute Error in Jacobi Integral and Limit $t_s$ for Numerical Solution of Total Equations

$$\rho = .(\rho^2)$$

$$r = (r^2)^{.5}$$

$$V^2 = \dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2$$

$$\dot{\rho} = (V^2)^{1/2}$$

$$\begin{aligned}\Delta C_c &= V^2 + 2[\epsilon_1(\dot{x}'y' - \dot{y}'x') - \frac{1}{\rho} - C_8(x'C_{et} + y'S_{et}) - C_{part}] \\ &+ 2C_8 \frac{[2(x'C_{et} + y'S_{et} - x_o) + \epsilon_2(\rho^2 - x_o^2)]}{rr_o(r_o + r)}\end{aligned}\quad (16)$$

where  $C_8$  is computed in Section 4.4 and  $C_{et}$  and  $S_{et}$  are computed in Section 4.25.

$$\frac{\Delta C_c}{C} = \frac{\Delta C_c}{C_{init}}$$

$$T_s = T_s \text{ modulo } P_{ss}$$

Go to Step 4.15 for solution of Kepler's equation.

#### 4.28.2 Compute the Fixed Orbit Values

$$x'_f = \rho_s C_v$$

$$y'_f = \rho_s S_v C_i$$

$$z'_f = \rho_s S_v S_i$$

$$\dot{x}'_f = -S_v/a_e^{1/2} \quad (127)$$

$$\dot{y}'_f = \frac{(C_v - e_s)}{a_e e_s^{1/2}} C_i$$

$$\dot{z}'_f = \frac{(C_v - e_s)}{a_e e_s^{1/2}} S_i$$

$$\rho_f^2 = x_f'^2 + y_f'^2 + z_f'^2$$

$$\rho_f = (\rho_f^2)^{1/2}$$

$$r_f = \{ 1 + \epsilon_2 [2(x_f' c_{\epsilon t} + y_f' s_{\epsilon t}) + \epsilon_2 \rho_f^2] \}^{1/2}$$

$$\begin{aligned} \Delta C_f &= \dot{x}_f'^2 + \dot{y}_f'^2 + \dot{z}_f'^2 + 2[\epsilon_1(\dot{x}_f' y_f' - \dot{y}_f' x_f') - \frac{1}{\rho_f}] \\ &\quad - C_8(x_f' c_{\epsilon t} + y_f' s_{\epsilon t}) - C_{\text{part}} \\ &\quad + 2C_8 \frac{[2(x_f' c_{\epsilon t} + y_f' s_{\epsilon t} - x_o) + \epsilon_2(\rho_f^2 - x_o^2)]}{r_f r_o(r_o + r_f)} \end{aligned} \quad (16)$$

where  $C_8$  is computed in Section 4.4 and  $c_{\epsilon t}$  and  $s_{\epsilon t}$  are computed in Section 4.25.

$$IT = 13 \text{ ITP} - 12$$

$$\frac{\Delta C_f}{C} = \frac{\Delta C_f}{C_{\text{init}}}$$

$$\Delta \rho_f = \frac{1}{a} [(x_f' - a_p(IT+2))^2 + (y_f' - a_p(IT+3))^2 + (z_f' - a_p(IT+7))^2]^{1/2}$$

$$\begin{aligned} \dot{\Delta \rho}_f &= a^{1/2}[(\dot{x}_f' - a_p(IT+8))^2 + (\dot{y}_f' - a_p(IT+11))^2 \\ &\quad + (\dot{z}_f' - a_p(IT+12))^2]^{1/2} \end{aligned}$$

$$\begin{aligned} \Delta \rho_c &= [(x' - a_p(IT+2))^2 + (y' - a_p(IT+3))^2 \\ &\quad + (z' - a_p(IT+7))^2]^{1/2}/a \end{aligned}$$

$$\begin{aligned} \dot{\Delta \rho}_c &= a^{1/2}[(\dot{x}' - a_p(IT+8))^2 + (\dot{y}' - a_p(IT+11))^2 \\ &\quad + (\dot{z}' - a_p(IT+12))^2]^{1/2} \end{aligned}$$

#### 4.28.3 Set Quantities Just Computed Equal to Values in Plot Array

$$a_p(IT + 2) = |\Delta C_c/C|$$

$$a_p(IT + 3) = |\Delta C_f/C|$$

$$a_p(IT + 7) = \Delta \rho_c$$

$$a_p(IT + 8) = \Delta \rho_f$$

$$a_p(IT + 11) = \Delta \dot{\rho}_c$$

$$a_p(IT + 12) = \Delta \dot{\rho}_f$$

#### 4.29 TEST VALUE OF HALT FLAG, KHALT

If KHALT = 3, go to Step 4.20.

If KHALT = 2, go to Step 4.14.1.

If KHALT = 1, continue below.

#### 4.30 TEST, IF PRINTING IS DESIRED

If  $W_{13} \leq 0$ , go to Step 4.22.

If  $W_{13} > 0$ , go to Step 4.20.

#### 4.31 CALL THE PLOT SUBROUTINE

The arguments for subroutine PLOT are:

$a^{3/2}$

$T_f$

a

e

i°

μ

c<sub>init</sub>

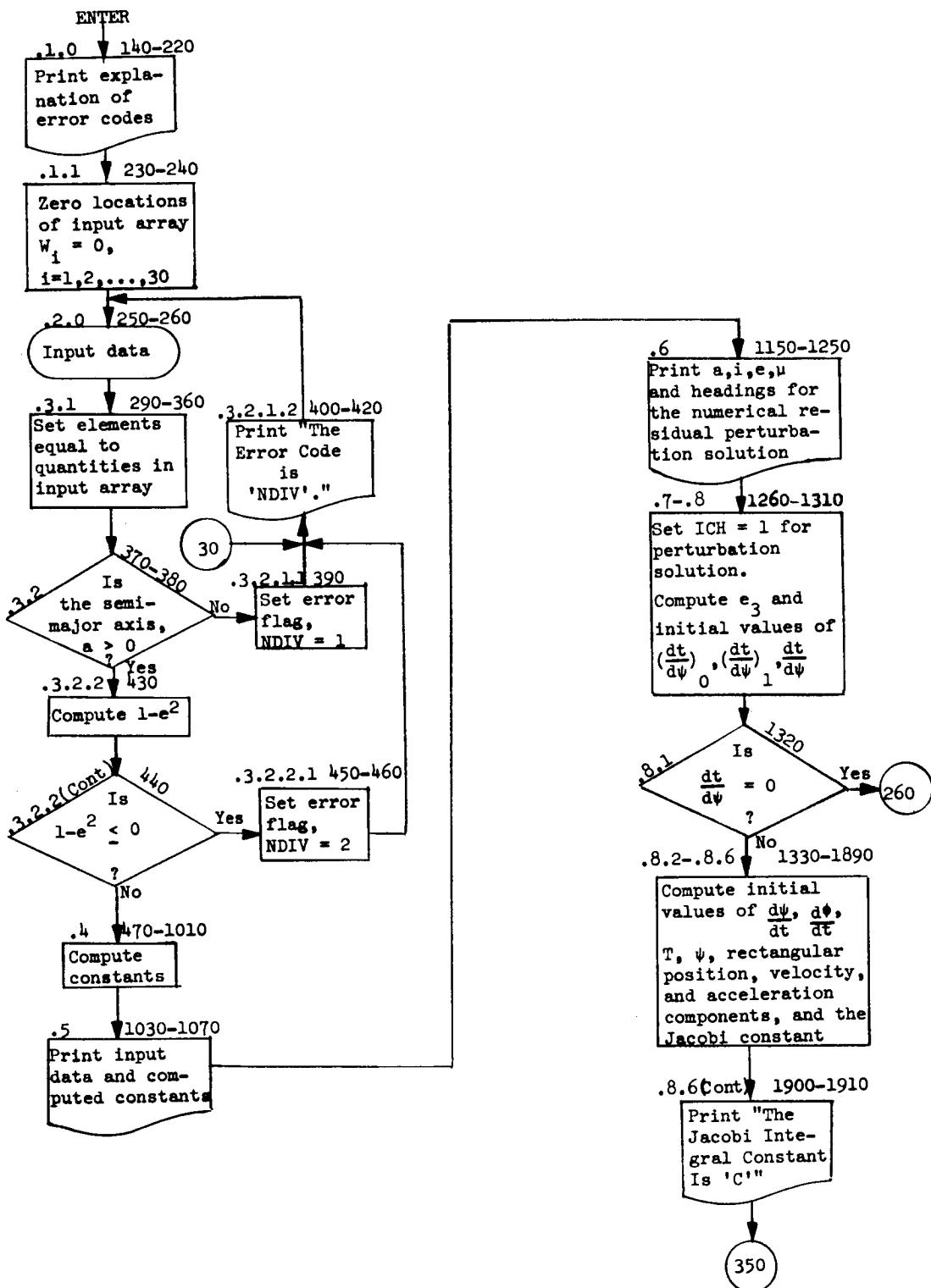
P<sub>ss</sub>

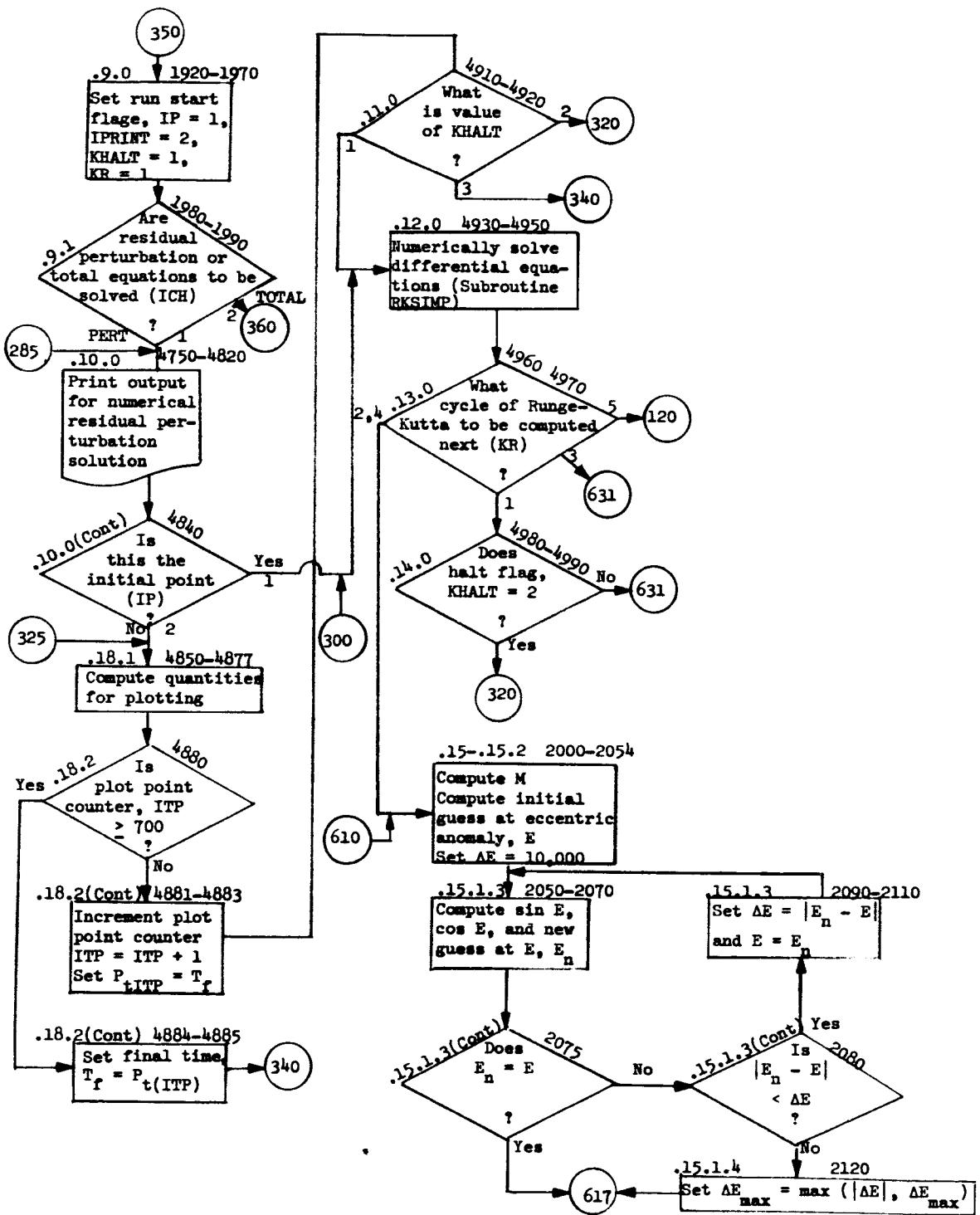
4.32 RUN IS COMPLETED, CALL THE NEXT CASE AT STEP 4.2

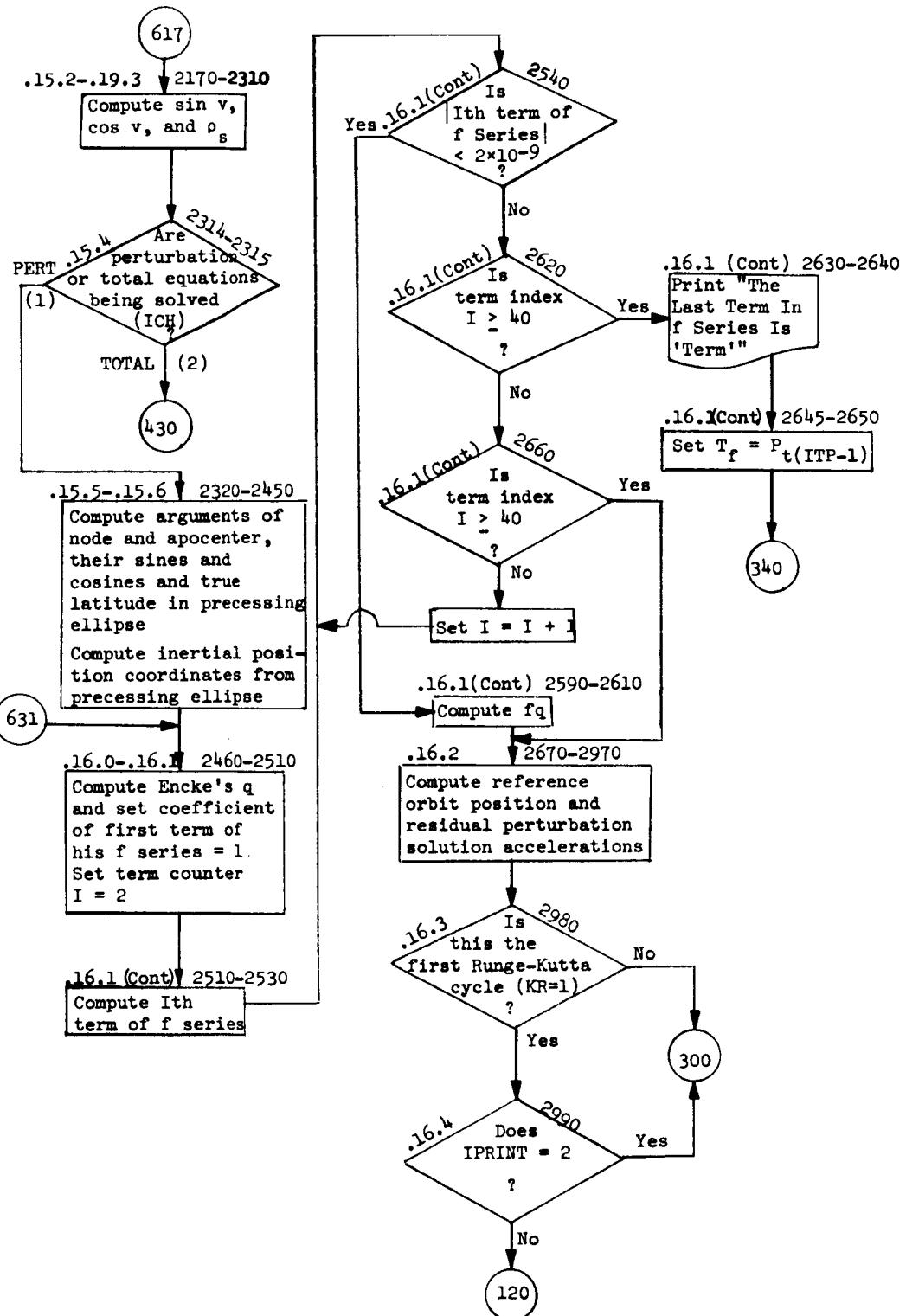
Section 5  
MAIN PROGRAM DETAIL FLOW CHARTS

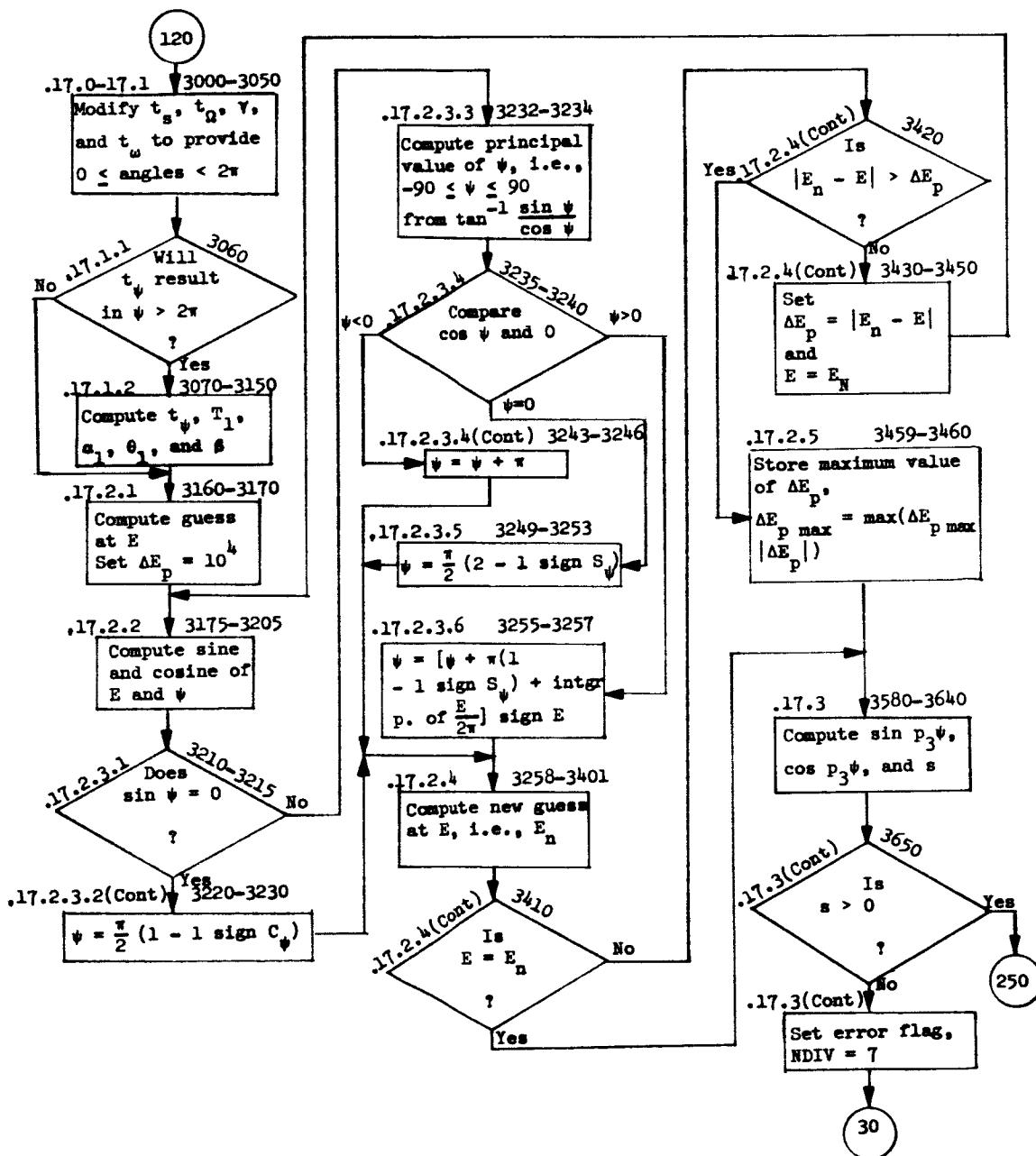
**5.1 EXPLANATION**

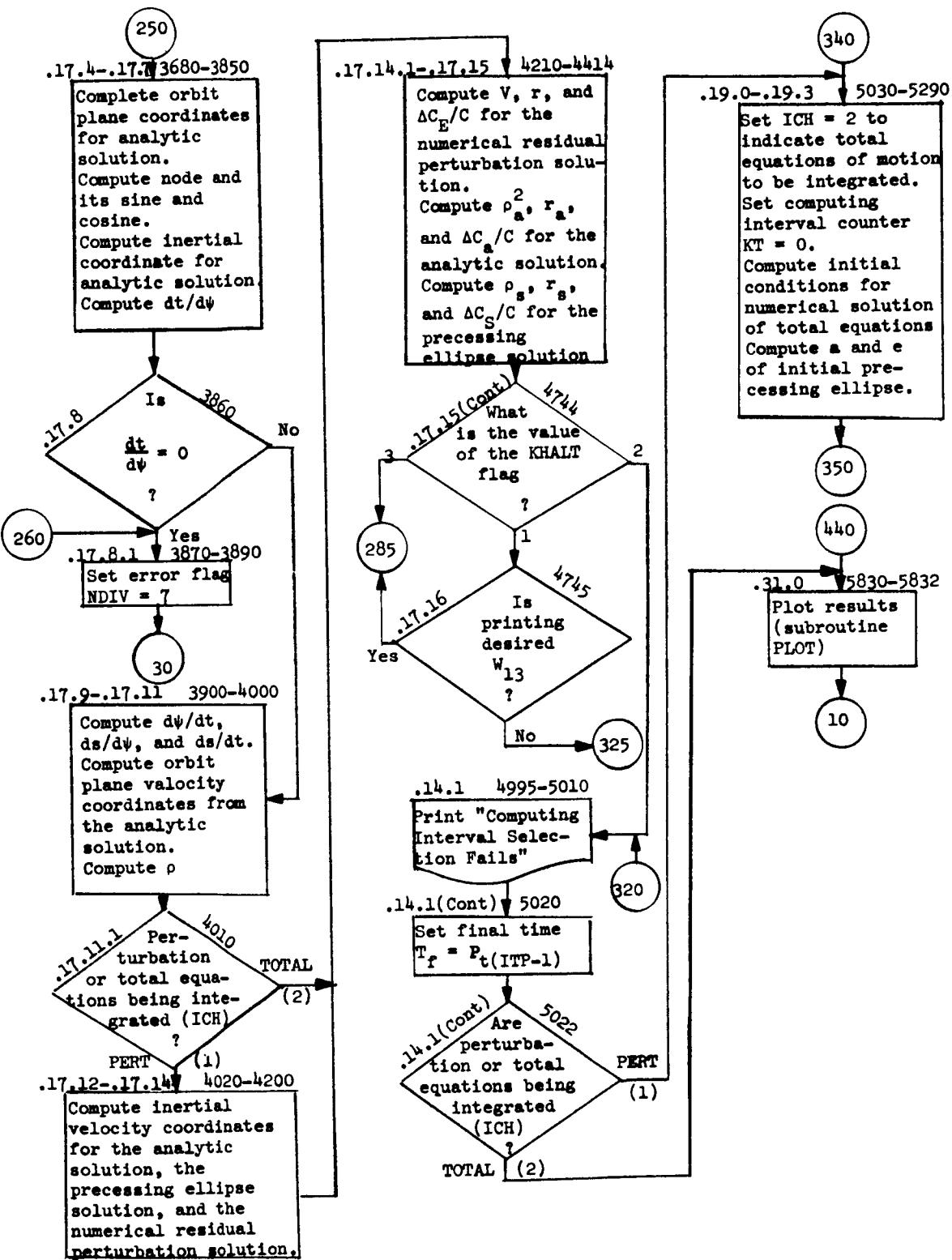
The flow charts of this section describe the logic of the main program completely, i.e., all conditional transfers are shown. The numbers to the top left at each box are the subsection numbers of Section 4 where the operations mentioned in the box are detailed. The numbers on the top right are the card numbers of the program listing, Section 6.

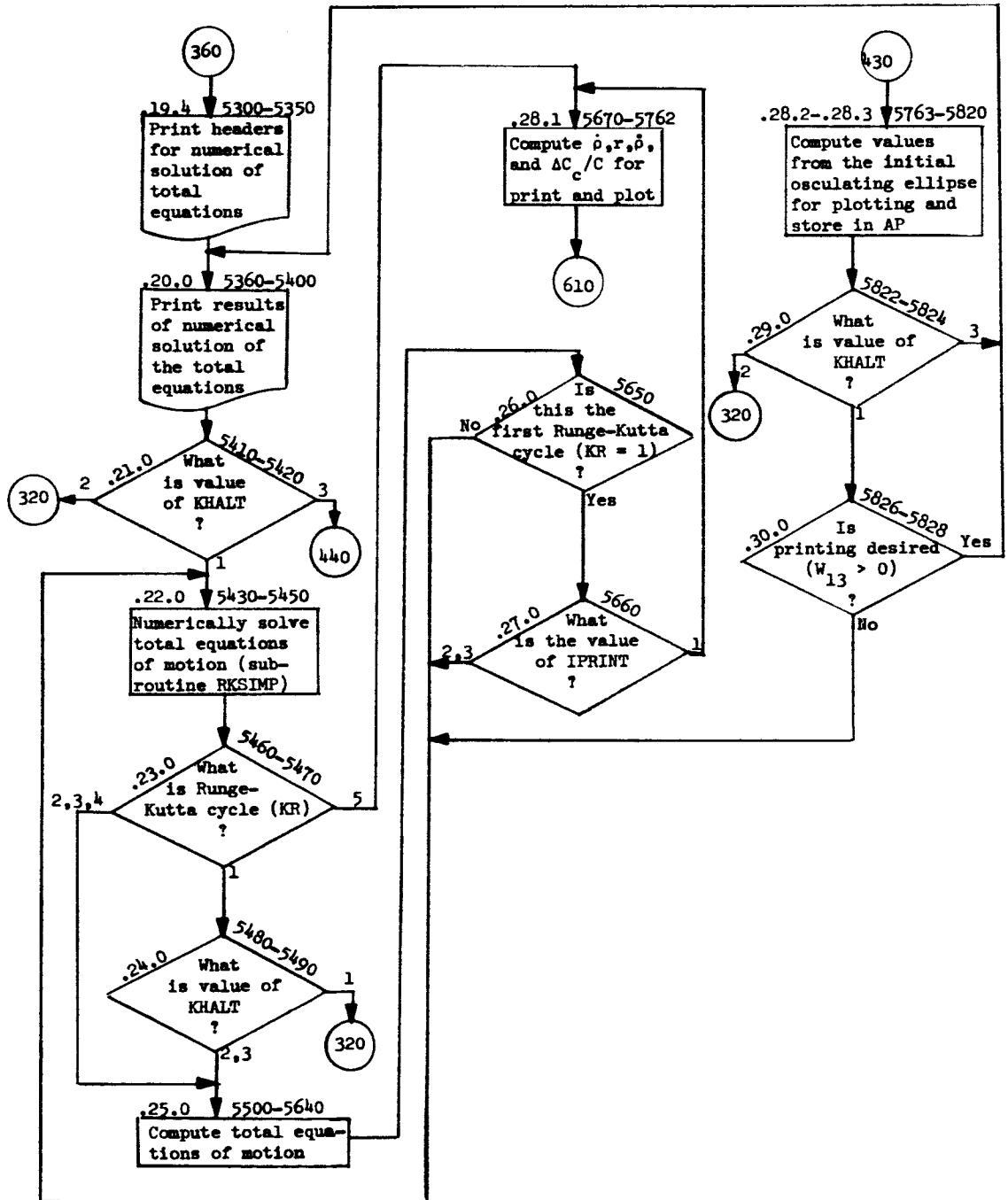












## Section 6

### MAIN PROGRAM LISTING

This section contains the listing of the main program. The subsection numbers on the comment cards refer to the subsections of Section 4, Equations in Order of Solution, wherein the code is explained.

M144 JFRRY EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)

```

09/01/65

C NUMERICAL RESIDUAL PERTURBATION SOLUTIONS APPLIED TO THE PROBLEM 0070H144
C OF A CLOSE SATELLITE OF THE SMALLER BODY IN THE RESTRICTED THREE- 0080H144
C BODY PROBLEM 0085H144
DIMENSION M(30), WR(30), PSI(25), FPS(25) 0090H144
CIMFESN ECS(25), TERM(40), TNUM(40), TDEN(40), 0100H144
1 FRA(40), FAC(40), PTINE(700), API(9100) 0110H144
COMMON W, APRIN, TS, TPSI, TNOU, TPER, GAMMA, EPI, 0120H144
1 ITP, PTIME, AP 0130H144
C1.0 PRIN1 EXPLANATION OF ERROR CODE 0140H144
WRITE(6, 901) 0150H144
901 FORMAT(1H1.3CH ERROR CODE REASON FOR HALT//4X2H 1 BX20H SEMI-MO160H144
IJNR AXIS = 0/4X2H 2.8X3H ECCENTRICITY EQUALS OR EXCEEDS 1/ 0170H144
5 4XH 7.8333H S (THE APPROX RADIUS VECTOR) = 0/4X2H 8.0X31H0210H144
6 DERIVATIVE OF TIME WRT PSI = 0) 0220H144
C1.1 SFT INPUT ARRAYS = 0 0230H144
DO 7 I = 1, 30 0240H144
7 W(I) = 0. 0245H144
KKKK = 1 0250H144
C 2.0 PROGRAM INPUT 0260H144
10 CALL INPUT (W(1), W(30), WR(1)) 0280H144
KFAIL = 0 0285H144
C3.0 MOVE INPUT TO WORKING STORAGE AND CHECK INPUT 0290H144
C3.1 MOVE INPUT TO WORKING STORAGE AND FORM NECESSARY CONSTANTS 0300H144
A1 = ABS (W(1)) 0310H144
A2 = A1 ** 2 0320H144
AS = A1 0330H144
FC1 = ABS (W(2)) 0340H144
ES = FC1 0350H144
DFGI = W(3) 0355H144
TF = W(7) 0360H144
FP4 = ABS (W(4)) 0370H144
C3.2 CHECK ON CONSTANT DIVISORS. HALT RUN IF ANY ARE ZERO 0375H144
C3.2.1 TEST SEMI MAJOR AXIS 0380H144
IF (A1) 20, 20, 50 0385H144
C3.2.1. SET ERROR CODE = 1 0390H144
20 NDIV = 1 0395H144
C3.2.1.2 PRINT ERROR CODE AND GO TO STEP 2.0 0400H144
30 WRITE(6, 40) NDIV 0410H144
40 FORMAT(1H .1BH THE ERROR CODE IS 15) 0420H144
GO TO 10 0425H144
C3.2.2 COMPUTE ANC TEST E SUB 2 0430H144
50 EC2 = 1. - EC1 ** 2 0440H144
IF (FC2) 60, 60, 70 0445H144
C3.2.2.1 SET ERROR CODE TO 2 AND GO TO STEP 3.2.1.2 0450H144
60 NDIV = 2 0460H144
GO TO 30 0465H144
C4.0 COMPUTE CONSTANTS 0470H144
70 AEC = A1 * EC2 0480H144
AESQ = SQRT (AEC) 0490H144
FC2 = SQRT (FC2) 0500H144
A1 = A1 * A1 * A1 0510H144
EP2 = SQRT (EP4) 0520H144
TNUM(1) = 3. 0530H144
DO AC1 = 1, 33 0540H144
DO AC1 = 2, 40 034

```

| M144 | JFRN4  | EXTERNAL FORMULA NUMBER  | SOURCE STATEMENT | INTERNAL FORMULA NUMBER(S) | 09/01/65 |
|------|--|--------------------------|------------------|----------------------------|----------|
|      |  | TNUM(1) = TNUM(1-1) + 2. |                  | 0550H144                   | .35      |
| 60   | IENM(1)  | = (IENM(1-1) + 1.        |                  | 0560H144                   | .36      |
|      | ITP = 2  |                          |                  | 0561H144                   | .37      |
|      | PTIM(1) = 0.   |                          |                  | 0562H144                   | .38      |
|      | DN(91) = 1..13                                       |                          |                  | 0563H144                   | .39      |
| 91   | AP(1) = 0.   |                          |                  | 0564H144                   | .40      |
| 90   | A32 = SORT(A33)                                      |                          |                  | 0565H144                   | .41      |
|      | A92 = A32 * e1_3                                     |                          |                  | 0570H144                   | .42      |
|      | A532 = A32   |                          |                  | 0580H144                   | .43      |
|      | EPA1 = SORT(FP2)                                     |                          |                  | 0590H144                   | .44      |
|      | C1 = 3.15 * EP1 * A3 * EC1                           |                          |                  | 0600H144                   | .45      |
|      | FPA = EP1 * A32                                      |                          |                  | 0610H144                   | .46      |
|      | GNUM1 = (-A12) * (1.24 * EC1 + 11.1/12.              |                          |                  | 0620H144                   | .47      |
|      | F1M1 = (-.75) * A32                                  |                          |                  | 0630H144                   | .48      |
|      | REFAL M1   |                          |                  | 0640H144                   | .49      |
|      | A1 = - A32 * (1. + 12. * EC1) / 6.                   |                          |                  | 0650H144                   | .50      |
|      | DNFGL = 43.01 - 7. * (26. * EC1)/12.                 |                          |                  | 0660H144                   | .51      |
|      | P1 = 1. / (1. + EP2 * ONEGI) - EPA *                 |                          |                  | 0670H144                   | .52      |
| 1    | 2. * ELMR1)  |                          |                  | 0680H144                   |          |
| 110  | RAD = DEG 1 / 57.2953745                             |                          |                  | 0690H144                   | .53      |
|      | C1 = COS(RAD 1)                                      |                          |                  | 0710H144                   | .54      |
|      | SI = SIN(RAD 1)                                      |                          |                  | 0720H144                   | .55      |
|      | AFA1 IS  |                          |                  | 0730H144                   | .56      |
|      | IS = RAD1  |                          |                  | 0750H144                   | .57      |
|      | CIS = COS(LIS)                                       |                          |                  | 0760H144                   | .58      |
|      | SIS = SIN(LIS)                                       |                          |                  | 0770H144                   | .59      |
| 602  | DPFR = (1.-5 * EP2 * A32 - EPA * QMEGL * GNUM1) /    |                          |                  | 0780H144                   |          |
|      | (1.1. + EP2 * QMEGL) * EPA                           |                          |                  | 0800H144                   |          |
|      | DNOD = DNOD * DPER                                   |                          |                  | 0810H144                   |          |
|      | DPER = DNOD * 2 * DPER * 2                           |                          |                  | 0820H144                   |          |
|      | DPNAC = DNOD * 2                                     |                          |                  | 0830H144                   |          |
|      | A12 = SORT(A11)                                      |                          |                  | 0840H144                   |          |
|      | P1 = 1. / (1. + EP2 * QMEGL) - EPA *                 |                          |                  | 0850H144                   |          |
|      | C2 = 1.875 * EP1 * A12 * EC1                         |                          |                  | 0860H144                   |          |
|      | C3 = .375 * FPI * RAU1 * A1 * A32 / (1. - EPA * A32) |                          |                  | 0870H144                   |          |
|      | CA = DNOD  |                          |                  | 0880H144                   |          |
|      | C7 = C6 * C6   |                          |                  | 0890H144                   |          |
|      | CA = 1. - FP4  |                          |                  | 0900H144                   |          |
|      | C9 = AEC * A12 * EC2                                 |                          |                  | 0910H144                   |          |
|      | C10 = FP2 * QMEGL / (1. + EP2 * QMEGL) + EPA *       |                          |                  | 0920H144                   |          |
|      | (1. + FPI * M1)                                      |                          |                  | 0930H144                   |          |
|      | C11 = 2. * P1 - 1.                                   |                          |                  | 0940H144                   |          |
|      | C12 = 2.* DNOD / AESQ                                |                          |                  | 0950H144                   |          |
|      | C13 = 2.* DPFR / AESQ                                |                          |                  | 0960H144                   |          |
|      | C14 = 2.* DPDN                                       |                          |                  | 0961H144                   |          |
|      | C15 = -C12 - C13 * C1                                |                          |                  | 0962H144                   |          |
|      | C16 = -C14 - OPNSO * C1                              |                          |                  | 0963H144                   |          |
|      | C17 = C12 * C1 * C13                                 |                          |                  |                            |          |
|      | C18 = C14 * C1 + OPNSO                               |                          |                  |                            |          |
|      | C19 = DPFR * DPFR * SI                               |                          |                  |                            |          |
|      | C20 = EC1 * S1 * C13                                 |                          |                  |                            |          |
|      | T1 = 6.2931853 * A32 + C1 * SIN(6.2831853 * C11)     |                          |                  |                            |          |
|      | 1. - A57 * 1.375 * SIN(6.2831853 * 2. * P1) * EP2    |                          |                  |                            |          |
|      | C21 = S1 * C13                                       |                          |                  |                            |          |
|      | C22 = EC1 * C15                                      |                          |                  |                            |          |
|      | C23 = EC1 * C17                                      |                          |                  |                            |          |

```

H144 JFRRY EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)
      C24 = SINT (1. - ES ** 2)
      PRS = 6.281853 * A3
      PNOD = 6.2831853 / OPER
      PPFR = 6.2831853 / OPER
      DFM = 0.
      DEPM = 0.
      ALPHI = 0.
      THETAI = 0.
      PH11 = 0.
      CALPA = J.
      SALPA = 0.
      STIFIA = 0.
      CTFIA = 0.
      PRINT CONSTANTS
      WRITE (6, 1) (A11), I = 1.151, OMEGI, GNUL, ELMB, PI.
      1 P1, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, C11, EC2
      2 P11, DNDQ, DPDR, DPNSQ
      2 * OPER.

      1 /4F0.8/
      1 PRINT HEADER FOR NUMERICAL RESIDUAL PERTURBATION SOLUTION LISTING
      1 6.0 THIN INPUT VALUES OF A.E., AND MU
      117 WRIT 10, 5021 A1, OEG 1, ECI, EP4
      902 FORMAT(1H48 THE INITIAL CONDITIONS ARE SEMI-MAJOR AXIS E17.6/31X14H INCLINATION-DEG E17.6/31X14H ECCENTRICITY E20.8/4X12H MASS1170H144
      2 RATIO E2.2)
      903 FORMAT(1H6/6L TIME DELTA X
      17 RHO/5MH DELTA Y
      2 RHO DOT/53H C APPROX DELTA Z, DELTA Z DOT, DELTA Y DOT, DELTA X DOT APPROX/51H
      3 DOT/51H X DOT X APPROX Y DOT Y APPROX Z DOT Z APPROX/51H C EXACT Z
      4 Y Z DOT APPROX Z DOT APPROX)
      7.0 SFT PERTURBATION-TOTAL FLAG
      1CH * 1
      COMPUTE INITIAL VALUES OF POSITION, VELOCITY, AND ACCELERATION
      8.0
      FCA = 1. + ECI
      DROPS = A32 * EC3 ** 1.5 / SQRT (1. - EC1)
      OTDOPS = 7./12. * EP2 * A92
      OTDOPS1 = OTDOPS + OTDOPS
      8.1 FIRST THE VALUE OF THE DERIVATIVE OF TIME WITH RESPECT TO PSI
      IF (OTDOPS1 .LT. 119.) 260, 119
      2 COMPUTE THE DERIVATIVE OF PHI WITH RESPECT TO T AND SET INITIAL
      VALUES OF T AND PSI
      119 OTDOP = 1. / OTDOPS
      OPHDIT = DPS1DT / (1.+EP2 * OMEGI) - EP2 * GNUL
      T = C.

      8.3 POSITION COORDINATES
      X2P = A1 * EC3
      Y2P = 0.
      Z2P = 0.
      XP = X2P
      YP = 0.
      ZP = 0.
      X = XP

```

H144 JFRRV EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)

```

Y = C.
Z = C.
DX = 0.
CY = 0.
D7 = Q.
D7 = Q.
RHO = X
A = 1. + FP2 * X
RCU = (1. + EP2 * X) ** 3
XS = XP
YS = Q.
ZS = Q.
XSD = 0.
YD01 = - (DTODPS + FP2 * DMFC1 + DTODPS * (1. + EP2 * OMEG1))
      / (DTODPS * (1. + EP2 * OMEG1) + DTODPSI - EP2 * GMN1-OPER)
DYN = X * GDD1 + CI
DZ0 = X * GDD1 + SI
AT00 = SORT (YD ** 2 + ZD ** 2)
XSD = 0.
YSD = X * (DND0 + (1. / DTODPS + OPER) * CI)
ZSD = X * (1. / DTODPS + OPER) * SI
C 8.5 ACCELERATION COMPONENTS
FP2X = FP2 * X
DX2D = CB * EP2X * (2. + EP2X * (3. + EP2X)) / RCU
      1 + 2. + (OPER + DND0 * CI) * SORT((1. - ES) / AS / (1.
      2 + FS))
      + XS * (2. * DPDN * CI + DNSQ)
      DYZD = 0.
CZ20 = 0.

C 8.6 INITIAL VALUE OF JACOBI CONSTANT
ACDR1 = X ** 2 * (DPHID1 ** 2 + C7 ** -2 * (C6 * EPI -
      1 DPHID1 * (C6 - EPL1 * CII) - 2. * (1. / X + C8 * X))
      RFAL JACUBI
JACUBI = ACDBI - 2. * C8 * EP2 / (1. + EP2 * X)
XNAUT = X
RHAUT = 1. + EP2 * X
YACDR1 = 0.
YACDP = 0.
WRITE (6, 351) JACORI
351 FORMAT (1HO.32H THE JACOBI INTEGRAL CONSTANT IS E17.8)
C9.0 SET RUN START FLAGS
350 IP = 1
      IPRINT = 2
      KHALI = 1
      KR = 1
      TFSI PERTURBATION-TOTAL FLAG
      GO TO (789, 360). ICH

```

H164 JERRY FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S) 09/01/75  
 EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)

```

C15.0 COMPUTE TWO-BODY EQUATIONS
C15.1 SOLVE KEPFR'S EQUATION
C15.1.1 FORM CONSTANTS
C16.0 TAS32 = TS / AS32
C16.1 TAS32 - ES * SIN(TAS32)*((1.+ES*cOS(TAS32))
      ES = TAS32 - ES * SIN(TAS32)*((1.+ES*cOS(TAS32))
      DT = 10000.
      DT = 10000.

C15.1.3 COMPUTE NEW GUESS AT E
616 SESA = SIN(ESAA)
617 CESA = COS(ESAA)
618 SESA = SIN(ESAA)
619 CESA = COS(ESAA)
620 DEM = AMAX1( DEM, ABS(DLTE))
621 SIV = C24 * SESA / DEN
617 SIV = C24 * SESA / DEN
618 COV = (CESA + ES) / DEN
619 COMP1(RHO, SUB, S)
620 SNOF = AS * DEN
623 RHOS = AS * DEN
C15.4 TFSI PERTURBATION-TOTAL FLAG
GO TO (431, 43C), ICH
C15.5 COMPUTE ARGUMENTS OF NODE AND PERIGEE AND THEIR SINES AND COSINES
431 SNOF = DNOD * TNOD
SPER = OPER * TRER
CSNO = COS(INSODE)
SSNO = SIN(INSODE)
CUPER = COS(SPER)
SIPER = SIN(SPER)
STRL = SIV * CUPER + COV * SIPER
CTR = COV * CCPER - SIV * SIPER
C15.6 COMPUTE THE PRECESSIONING TWO BODY POSITION COORDINATE IN THE
C INERTIAL COORDINATE SYSTEM
XS = RHOS * (CTRL * CSNO - STRL * SSNO * CISI)
YS = RHOS * (CTRL * SSNO + STRL * CSNO * CISI)
ZS = RHOS * STRL * SIS
SFPT = SIN(GAMMA)
CPFT = COS(GAMMA)
C16.0 ACCIFICATION COMPONENTS FOR NUMERICAL RESIDUAL PERTURBATION
C EQUATIONS OF MOTION
C16.1 COMPUTATION OF THE SMALL DIFFERENCES BETWEEN THE TWO BODY TERMS
631 O = ((XS + DX / 2.) * DX + (YS + DY / 2.) * DY +
      ((ZS + DZ / 2.) * DZ + (OZ / 2.) * RHO) ** 2
      TFRM11) = 1.
      ON 625 I = 2..40
      FRA(I) = Q * TFRM11 / TDEN11
      TFRM11 = TFRM11 - 1. * FRA(I)
      IF (ARS(TFRM11)-2.E-91624.6224. 622
          FAC(I) = 1. - FRA(I)
          L = I - 1
          ON 625 J = 2..L
          K=I-J
          625 FAC(I)=1.-FRA(K+1)*FAC(J-1)
  
```

W164 JERRY FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)

|  |  |          |
|--|--|----------|
| F0 = 3.0 * Q * FACIL   |  | 09/01/65 |
| GO TO 630  |  |          |
| 626 IF (I - 40) .GT. 629, 627, 627   |  |          |
| 627 WRITF (6, 628) TERM11  |  |          |
| 628 FORMAT(IHO,29H THE LAST TERM IN F SERIES IS E17.8)                       |  |          |
| RF = P1MF (TIP - 1)  |  |          |
| TIP = 1  |  |          |
| GO TO 340  |  |          |
| 629 CONTINUEF  |  |          |
| C16.2 COMPUTE NUMERICAL RESIDUAL PERTURBATION ACCELERATION COMPONENTS        |  |          |
| 630 X = X5 + DX  |  |          |
| Y = Y5 + DY  |  |          |
| Z = Z5 + DZ  |  |          |
| RHOSC = X*X + Y*Y + Z*Z  |  |          |
| RSG = 1. + EP2 * (2. + IX * CEPT + Y * SEPT) + EP2 * RHOSQ1                  |  |          |
| R = SQRT (RSG)   |  |          |
| RCU = RSG * R  |  |          |
| FPR1 = CA * FP2 / RCU  |  |          |
| FPR2 = CB * EP2 * (2. + (X * CEPT + Y * SEPT) + EP2 *                        |  |          |
| 1 RHOSQ1 * (I1. + K * (I1. + R1)) / RCU / (I1. + R1)                         |  |          |
| RHOSCU = RHOS * 3  |  |          |
| R1 = C15 + C16 * RHOS  |  |          |
| R2 = C17 + C18 * RHOS  |  |          |
| 0x20 = (FO * X - DX) / RHOSCU - EPRL * X + EPRL * CEP                        |  |          |
| 1+5IRL * SSNO * B1 + CTRL * CSNO * B2 - SIPR * SSNO * C22 - COPR * 2810H144  |  |          |
| 2 * CSNO * C23   |  |          |
| DY20 = (FG * Y - DY) / RHOSCU - EPRL * Y + EPRL * SEPT                       |  |          |
| 1 - STRL * CSNO * B1 + CIRL * SSNO * B2 + SIPR * CSNO * C22 - COPR2881H144   |  |          |
| 2 * SSNO * C23   |  |          |
| 0720 = (FO * Z - DZ) / RHOSCU - EPRL * Z                                     |  |          |
| 1 + SIRL * IC21 + RHOS * C19 * -SIPR * C20                                   |  |          |
| C16.3 TEST THF RUNGF-KUTTA FLAG. KR  |  |          |
| GO TO (121, 300, 300, 300).KR  |  |          |
| C16.4 TEST THE PRINT FLAG. IPRINT  |  |          |
| 121 GO TO (1120, 30C, 120). IPRINT   |  |          |
| C17.0 COMPUTATIONS FOR PRINT ONLY  |  |          |
| C17.1 REDUCE ALL ANGLES THAT INCREASE WITH TIME BY INTEGER MULTIPLES OF 2 PI |  |          |
| 120 IS = AMOD (IS, PSS)  |  |          |
| TNOD = AMOD (TNOD, PNOD)   |  |          |
| TPFR = AMOD (TPER, PPER)   |  |          |
| GAMMA = AMOD (GAMMA, 6.2831853)  |  |          |
| C17.1.1 TEST VALUE OF 1 SUB PSI  |  |          |
| IF (TPSI - 111007). 632,   |  |          |
| C17.1.2 SF1 T SUB PSI  |  |          |
| 637 TPSI = TPXI - T1   |  |          |
| PH1=AMOD(PHI1+6.2831853 / (1.+ EP2 * QNEG1) - EP2 * GNUL * T1.               |  |          |
| 1.6.2831853)   |  |          |
| ALPHAI = AMOD( ALPHAI + ALPHAI + 6.2831853 * PI. 6.2831853 )                 |  |          |
| 1 + 6.2831853 * PI. 6.2831853 )  |  |          |
| CALPHA = CCS (ALPHAI)  |  |          |
| SALPHA = SIN (ALPHAI)  |  |          |
| ALPHAI2 = AMOD( ALPHAI )   |  |          |
| 1 + 6.2831853 * PI. 6.2831853 )  |  |          |
| THEET1 = AMOD (THEET1 + 6.2831853 * C11. 6.2831853 )                         |  |          |
| SINETA = SIN (INETA)   |  |          |

H144 JFRRY EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S) 09/01/65

```

C17.FIA = COS (THEIA)
THFIA2 = APGC (THEIA) + 6.2831853 * C11, 6.2831853)
T1 = 6.2831853 * A32 + C1 * (SIN (THEIA2) - SIN (THEIA))
1 - 1.375 * A92 * EP2 * (SIN (ALPHA2) - SIN (ALPHA1))
BETA = AMOD (BETA + 6.2831853 * P3, 6.2831853)

C17.2 SOLUTION OF THE PERTURBED KEPLER EQUATION
C 17.2.1 COMPUTE INITIAL GUESS AT E
C 1007 FCAN = (TPSI/A32) - ECI * SIN(TPSI/A32) * (1. - ECI * COS(TPSI/A32) / 1)
1) DLTFP = 10000.

C 17.2.2 COMPUTE SINE AND COSINE OF PSI
  8 SEC = SIN (ECAN)
  CFC = COS (ECAN)
  DFN = 1. * ECI * CEC
  SPS = EC2 * SEC / DEN
  CPS = (CFC + ECI) / DEN

C 17.2.3 COMPUTE PSI
C 17.2.3.1 TFST SIN PSI
  IF (1. SPS) 150.125.150
C 17.2.3.2 SFT PSI = ZERO OR PI
  125 PSI = 1.5707963 * (1. - SIGN (1.0 * CPS))
  GO TO 3
  GN TC 3

C17.2.3.3 COMPUTE PRINCIPLE VALUE OF PSI
  150 PSI = ATAN (SPS/CPS)
C17.2.3.4 TFST COS PSI
  IF (CPS) 170.180.2
  170 PSI = PSI + 3.1415926
  GO TO 3
C17.2.3.5 SFT PSI = PI/2 OR 3PI/2
  180 PSI = 1.5707963 * (2. - SIGN(1.0 * SPS))
  GO TO 3

C17.2.3.6 PLACE PSI IN THE FIRST OR THIRD QUADRANT
  2 PSI = SIGN(PSI) + 3.14159265 * (1.0 * SIGN1.0 * SPS) + 2. * AINT (ECAN)
  1 / 6.28318531. ECAN
  C 17.2.4 COMPUTE NEW GUESS AT ECCENTRIC ANOMOLY
  3 CLIPS = C11 * PSI
  SCILIPS = SIN(CLIPS)
  CCCLIPS = COS(CLIPS)
  CPI = CCCLIPS * CTHETA + STHETA * SCCLIPS
  SPI = SCCLIPS * CTHETA + STHETA * CCCLIPS
  C2P1PS = CCCLIPS * CPS - SCCLIPS * SPS
  S2P1PS = SCCLIPS * CPS + CCCLIPS * SPS
  CPIPS = C2P1PS * CALPHA - S2P1PS * SALPHA
  SPIPS = S2P1PS * CALPHA + C2P1PS * SALPHA
  DT1OPS = -C1 * (CPS - CPI) + A92 * (3.333333 * CPS - 2.75 * CPIPS3310H144
  1) * EP2
  DT1OE = EC2 * DT1OPS / DEN
  TONE = - C1 * SPS - CPI * STHETA + EP2 * A92 * (3.333333 * SPS - 330H144
  1. 1.375 * (SP1PS - SALPHA)) * N = (A32 * ECI * ( ECAN * CEC - SEC) + ECAN*DT1DE+TPSI-TONE)/4
  1A32 * DEN * DT1DE
  IF 1 FN - ECAN 4. 235. 4
  4 IF (ARSLEN - ECAN) - DLTEPI 5. 6. 6
  5 DLTFP = ARSLEN - ECAN
  FCAN = EN
  GO TO 8

```

H144 JFRNY FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S) 09/01/65

C17.2<sup>5</sup> STORE MAXIMUM FINAL VALUE OF DELTA E PERTURBED  
6 DEPM = ANAX(1 DEPM, ABS(TOLEP))

C17.3 COMPUTE AUXILIARY FUNCTIONS OF PSI AND T  
735 CPS = COS (PSI \* PSIT) \* COS (BETA) - SIN (PSI \* PSIT) \*  
1 SIN (BETA) - SIN (PSI \* PSIT) \* COS (BETA) + COS (PSI \* PSIT) \*  
1 SIN (BETA)

S = (1 - ECI \* CPS) / AEC + C2 \* (CPS - CPS1)  
1 + A2 \* (17. - 5. \* CPS) / 3. + CPS1 \* EPZ

IF (S) 740. 240. 250

740 NOIV = 7

GO TO 30

C17.4 COMPUTE DOUBLE PRECISION COORDINATES  
750 CPS1 = CPS / (1. + EP2 \* ONEG) - EP2 \* GNL + TPS1 \*  
X2P \* (COS (PHI) \* COS (PHII)) - SIN (PHI) \* SIN (PHII) / S  
Y2P \* (SIN (PHI) \* COS (PHII)) + COS (PHI) \* SIN (PHII) / S

C17.5 COMPUTE NUNI  
750 NUNI = C3 \* (SP3 - P3 \* SP5)  
ANODE, C6 \* TNUO  
CNO \* COS (ARNGDE)

SNO = SIN (ARNGDE)

C17.6 COMPUTE APPROXIMATE COORDINATES IN PLANE OF PLANETS  
C SINGLE PRIME SYSTEM 6.0.6.2.6.3

AP = X2P \* CNO - Y2P \* SNO + CI \* Z2P \* SNO + SI  
YP = X2P \* CNO + Y2P \* CNO - CI \* Z2P \* CNO + SI  
ZP = Y2P \* CNO + Y2P \* CNO - CI \* Z2P \* CNO + SI

C17.7 COMPUTE TIME DERIVATIVES OF PSI, PHI AND S  
D10PSI = C9 / (1. (1. - ECI \* CPS) \* 2) - CI \*  
1 CPS - CPI1  
2 + APZ \* (10. / 3. \* CPS - 2.75 \* CP1PS) \* EPZ

C17.8 TEST THE VALUE OF THE DERIVATIVE OF TIME WITH RESPECT TO PSI  
IF (INT(D10PSI)) 270. 260. 270

C17.8.1 SET THE ERROR CODE = 8 AND GO TO STEP 3.2.1.2  
260 NUNI = 8  
GO TO 30

C17.9 COMPUTE TIME DERIVATIVES OF PHI AND S  
270 D10SIOT = 1. / D10PSI  
DPIHCT = D10SIOT / (1. + EP2 \* ONEG1) - EP2 \* GNL1  
GSDPS1 = ECI \* AEC \* C2 \* (SP3 - CPS1)  
1 + A2 \* (5. / 3. \* SP5 - 2. \* SP1PS1) \* EPZ  
DSDT = D10PSI \* D10SIOT

C17.10 COMPUTE APPROXIMATE VELOCITY COMPONENTS IN ORBIT PLANE  
800 X2P = - Y2P \* DPHIOT - X2P \* DSDT / S  
Y2P = X2P \* DPHIOT - Y2P \* DSDT / S  
Z2P = C3 \* (CP3 - CPS) \* DPSIOT

C17.11 COMPUTE RHO  
RHO = SQRT (RHO1)

C17.11.1 IF ICH = 1 GO TO 17.12. IF ICH = 2 GO TO 17.14.1  
GO TO 800. 801. ICH

C17.12 COMPUTE APPROXIMATE VELOCITY COMPONENTS IN INERTIAL SYSTEM  
800 X2P = C6 \* (-X2P \* SNO - Y2P \* CNOC1 + Z2P \* CNOS1)  
1 + X2OP \* CNO - Y2OP \* SNO \* CI + Z2P \* SNO \* SI  
Y2P = C6 \* X2P \* CNO - Y2P \* SNO \* CI + Z2P \* SNO \* SI  
1 + X2OP \* SNO + Y2OP \* CNO \* CI - Z2P \* CNO \* SI  
Z2P = SNO \* CI + Z2P \* CNO \* CI

C17.13 COMPUTE PRECESSING KEPLERIAN VELOCITY COMPONENTS IN INERTIAL



H144 EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER (\$)

```

325 PTIMF ((ITP) = 1
  IT = 13 * ITP - 12
  AP (IT) = ABS (YACAPP)
  AP (IT + 1) = ABS (YACASI)
  AP (IT + 2) = X
  AP (IT + 3) = Y
  AP (IT + 4) = ABS (YACB1)
  AP (IT + 5) = SORT ((XP - X) * * 2 + (YP - Y) * * 2
  AP (IT + 6) = SORT ((XS - X) * * 2 + (YS - Y) * * 2
  AP (IT + 7) = 2 / A1
  AP (IT + 8) = 2
  AP (IT + 9) = A12 * SORT ((XD - XD) * * 2 + (YDP -
  1 * YD) * * 2 + (ZDP - ZD) * * 2)
  AP (IT + 10) = A12 * SORT ((XD - XD) * * 2 + (YSQ -
  1 * YD) * * 2 + (ZSQ - ZD) * * 2)
  AP (IT + 11) = YD
  AP (IT + 12) = ZD
  C18.2 TFSI TMF VALUE OF ITP
  IF (ITP = 700) 326, 327.
  326 ITP = ITP + 1
  PTIMF ((ITP) = 1
  GN TR 290
  327 IF = PTIMF ((ITP)
  GO TO 340
  C11.0 TEST HALT FLAG. KHALT
  290 GN TR (300, 320, 340). KHALT
  C12.0 CALL NUMERICAL SOLUTION SUBROUTINE FOR RESIDUAL PERTURBATION
  C EQUATIONS
  C 300 CALL RSIMP (KHK, IP, KC, T, DT, ICH, DX, DY, DZ, DDX,
  1 DYO, DZD, DZD, DYD, DZD, DZD, KHALT, TPRINT, TF)
  C 13.0 TFSI RUNGF-KLTIA FLAG
  GO TO 1310, 610, 631, 610, 1201, KR
  C 14.0 TEST HALT FLAG FOR INTEGRATION ACCURACY ERROR
  310 GO TO 1631, 320, 631. KHALT
  C14.1 PRINT COMPUTING INTERVAL SELECTION FAILED
  320 WRITE (6, 330)
  330 FORMA(1100, 35H COMPUTING INTERVAL SELECTION FAILS)
  TF = PTIMF ((ITP - 1)
  60 TR (340, 44C). ICH
  C19.0 INITIATE NUMERICAL SOLUTION OF TOTAL EQUATIONS OF MOTION
  C19.1 SET PERTURBATION-TOTAL FLAG
  340 ICH = 2
  KT = 0
  C19.2 COMPUTE INITIAL CONDITIONS FOR NUMERICAL SOLUTION OF TOTAL
  C EQUATIONS OF MOTION
  X = A1 * EC3
  Y = C0
  Z = C0
  GAMMA = 0.
  AM0 = X
  T = C
  DTDPSI = A32 * EC3 * 1.5 / SORT ((1. - EC1)
  1 * AS2 * EP2 * 7.
  OPS101 = 1. / DTDPSI
  
```

JERRY EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)

|   | 09/01/65   | 09/01/65   |
|---|--|--|
| C0HINIT = OPSINIT / ( 1. + EP2 * QMEGL ) - EP2 * GNU1                                   | 5170H144<br>5180H144<br>5190H144<br>5200H144<br>5210H144<br>5220H144<br>5230H144<br>5240H144 | *394<br>*395<br>*396<br>*397<br>*398<br>*399<br>*400 |
| X0 = 0.   |  |  |
| Y0 = X * (OPHINIT * CI + C6)  |  |  |
| Z0 = X * OPHINIT * SI   |  |  |
| RHO0 = SQRT (YD ** 2 + LD ** 2)   |  |  |
| RCU = (1. + EP2 * X) ** 3   |  |  |
| EP2X = EP2 * X  |  |  |
| X2D = -1. / X ** 2 + C8 * EP2X * (2. + EP2X * (3. +                                     |  |  |
| 1. EP2X)) / RCU   |  |  |
| Y2D = 0.  |  |  |
| Z2D = 0.  |  |  |
| YACORI = 0.   |  |  |
| <b>C19.3 COMPUTE SEMI-MAJOR AXIS AND ECCENTRICITY OF INIT. OSC. ELLIPSE</b>             |  |  |
| AS = 1. / (2. / RHO - RHOD ** 2)  | 5281H144<br>5282H144   | *404<br>*405   |
| FS = X / AS - 1.  | 5283H144   | *406   |
| AS32 = AS ** 1.5  | 5284H144   | *407   |
| PSS = 6.2831853 * AS32  | 5285H144   | *408   |
| AESQ = SORT (AS * (1. - ES ** 2))   | 5286H144   | *409   |
| GO TO 350   |  |  |
| <b>C19.4 PRINT HEADER FOR NUMERICAL SOLUTION OF TOTAL EQUATIONS OF MOTION</b>           |  |  |
| C PRINT   | 5300H144   |  |
| 360 WRITE (6, 370)  | 5305H144   | *410   |
| 370 FORMAT(1H0, 1.37H INTEGRATION OF COUPLE EQUATIONS OF MOTION//6X5H TIME, 16X5320H144 | 5310H144   |  |
| 12H X, 1.17X6H X, 001.13X10H X DBL DOT14X4H RHO/2X2H Y, 1.X6H Y DOT, 1.35330H144        |  |  |
| 2X10H Y DOT, OCT, 1.2X8H RHO DOT/17H JACOBI CONSTANT, 10X2H Z, 1.17X6H Z D5340H144      |  |  |
| 30T, 1.1X10H Z DOT, DOT)  |  |  |
| C20.0 PRINT NUMERICAL SOLUTION OF TOTAL EQUATIONS OF MOTION DATA                        |  |  |
| 380 IT = 1.5 * ITP - 12   | 5360H144   | *413   |
| I10 = IT + 12   | 5363H144   | *414   |
| WRITE (6, 390) T, X0, XD, X2D, RHO, Y, YD, Y2D, RHOD, YACOBI, Z, 2D,                    | 5366H144   |  |
| 1 Z2D, (AP11), 1 = IT, IT0),  | 5370H144   |  |
|   | 5380H144   | *415   |
| 390 FORMAT(1H0, //16.8-E19.8-3E21.8-3E35.8-3E21.6/E16.6,                                |  |  |
| 1 F19.8- 2E21.8 /5F20.8/4E20.8/4E20.8)  | 5390H144   |  |
| C21.0 TEST THF HALT FLAG. KHALT   | 5400H144   |  |
| 400 GD TG (410, 320, 440). KHALT  | 5410H144   |  |
| C22.0 CALL NUMERICAL SOLUTION SUBROUTINE FOR TOTAL EQUATIONS OF MOTION                  | 5420H144   | *421   |
| 410 CALL RK5IMP (KCR, IP, KC, T, DT, ICH, X, Y, Z, XD, YD,                              | 5430H144   |  |
| 1 ZD, Y2D, YD, 72D, KHALT, TPRINT, TF)  | 5440H144   |  |
| C23.0 TFST THF KUNGF-KUTTA FLAG. KR   | 5450H144   |  |
| GD TG (420, 275, 275, 275, 283). KR   | 5460H144   | *422   |
| C24.0 TFST THF HALT FLAG. KHALT   | 5470H144   |  |
| 420 GD TG (275, 320, 275). KHALT  | 5480H144   | *423   |
| C25.0 COMPUTE TOTAL EQUATIONS OF MOTION   |  |  |
| 275 RHOSQ = X * X + Y * Y + Z * Z   | 5490H144   |  |
| RHOCL = RHOSQ ** 1.5  | 5500H144   | *424   |
| GAMMA = ARD (GAMMA, 6.2831853)  | 5510H144   | *425   |
| CEPT = COS (GAMMA)  | 5520H144   | *426   |
| SFPT = SIN (GAMMA)  | 5530H144   | *427   |
| RSD = 1. + EP4 * RHOSQ + 2. * EP2 * (X * CEPT + Y * SFPT)                               | 5540H144   | *428   |
| R = SORT (RSQ)  | 5550H144   | *429   |
| RCU = RSQ * R   | 5560H144   | *430   |
| FPR1 = CB * EP2 * (2. * (X * CEPT + Y * SFPT) + EP2 *                                   | 5570H144   | *431   |
| FPR2 = CB * EP2 * (2. * (X * CEPT + Y * SFPT) / RCU / (1. + R))                         | 5580H144   | *432   |
| 1 RHOSQ) * (1. + R * (1. + R)) / RCU / (1. + R)   | 5590H144   | *433   |
| X2D = - X / RHOCU - EPRI * X + EP2 * CEPT   | 5600H144   | *434   |
|   | 5610H144   | *435   |
|   | 5620H144   |  |

M144 JFRRY EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)

09/01/65

```

Y2D = - Y / RHOUC - EPRI * Y + EPR2 * SEPT      5630H144   .436
Y2D = - Z / RHOUC - EPRI * Z      5640H144
C26.0 TEST THE RUNGE-KUTTA FLAG, KW      5645H144   .437
GO TO 141. 410. 410. 410. 410. KR
C27.0 TEST THE PRINT FLAG, IPRINT      5655H144   .438
411 GO TO 1283. 410. 410. IPRINT
C28.0 COMPUTE DATA FOR PRINT AND PLOT ROUTINE
C28.1 COMPUTE ERROR IN JACOBI INTEGRAL AND LIMIT T SUB S FOR NUMERICAL
C SOLUTION OF TOTAL EQUATIONS      5670H144
283 RHO = SORT (RHOSQ)
R = SORT (RSQ)
VSO = XD **2 + YD ** 2 + ZD ** 2      5675H144   .439
RHON = SORT (VSO)
VACORI = VSO + 2 * (EP1 * (XD * Y - YD * X) - 1. / RHO
1 - CR + (X * CEPT + Y * SEPT) - AC0B1 + 2. * CB * (Z.
2 + (X * LFPT + Y * SEPT - XNAUT) + EP2 * (RHOSQ - XNAUT
3 ** 2)) / R / RNAUT / (RNAUT + R)
VACORI = VACORI / JACOBI
TS = AMOD (115. PSS)
GO TO 610
C28.2 COMPUTE THE FIXED ORBIT VALUES
430 XF = RHOS * COV      5700H144   .440
YF = RHOS * SIV * CI      5710H144   .441
ZF = RHOS * SIV * SI      5720H144   .442
XF0 = - SIV / AFSD      5730H144   .443
YFD = (COV - FS) / AESC * CI      5740H144   .444
ZFD = (COV - FS) / AESQ * SI      5750H144   .445
RHOFSQ = XF ** 2 + YF ** 2 + ZF ** 2      5755H144   .446
RHOF = SORT (RHOFSQ)
RF = SORT (L. + EP2 * (Z. * (XF * CEPT + YF * SEPT) +
1 FP) * RHOFSQ)
VACAF = XFD ** 2 + YFD ** 2 + ZFD ** 2 + 2. * (EP1 *
1 (XFC * YF - YFU * XF) - 1. / RHOF - C8 * 1XF * CEPT +
2 YF * SEPT) - AC0B1 + 2. * CB * (Z. * (XF * CEPT + YF *
3 * SEPT - XNAUT) + EP2 * (RHOF SQ - XNAUT ** 2)) / RF /
4 RNAUT / (RHOF SQ)
IT = 13 * ITP - 12
YACAF = YACAF / JACOBI
DRF = SORT ((XF - AP(LT + 2)) ** 2 + (YF - AP(LT +
1 11) ** 2 + (ZF - AP(LT + 7)) ** 2) / AI
DRDF = AL2 * SORT ((XFD - AP(LT + 8)) ** 2 + (YFD -
1 - AP(LT + 11)) ** 2 + (ZFD - AP(LT + 12)) ** 2)
DRC = SORT ((X - AP(LT + 2)) ** 2 + (Y - AP(LT +
1 3)) ** 2 + (Z - AP(LT + 7)) ** 2) / AI
DRDC = AL2 * SORT ((XD - AP(LT + 8)) ** 2 + (YD -
1 AP(LT + 11)) ** 2 + (ZD - AP(LT + 12)) ** 2)
C28.3 SFT QUANTITIES JUST COMPUTED EQUAL TO VALUES IN PLOT ARRAY
AP(LT + 2) = ABS (YACAF)
AP(LT + 3) = ABS (YACAF)
AP(LT + 7) = DRC
AP(LT + 8) = DRF
AP(LT + 11) = DRDC
AP(LT + 12) = DRDF
C29.0 TEST VALUES OF HALT FLAG, KHALT
GO TO (619, 320, 380). KHALT
C30.0 TEST IF PRINTING IS DESIRED

```

| EXTERNAL FORMULA NUMBER  | SOURCE STATEMENT | INTERNAL FORMULA NUMBER(S) |
|--|------------------|----------------------------|
|  | 09/01/65         |                            |
| C146 IF (M(13)) 410, 410, 380  |                  | 5828H144                   |
| C31.0 CALL THE PLOT SUBROUTINE   |                  | 5829H144                   |
| 440 CALL PLOT (A32,IF,A1,EGL,DEGI,EP4,JACOBI,PSS)                      |                  | *471                       |
| 440 CALL PLOT (A32,IF,A1,EGL,DEGI,EP4,JACOBI,PSS)                      |                  | 5831H144                   |
| 440 CALL PLOT (A32,IF,A1,EGL,DEGI,EP4,JACOBI,PSS)                      |                  | *472                       |
| WHITE (6,230) DEM,DEPM   |                  | 5832H144                   |
| 230 FORMAT(1HO16HMAXIMUM DELTA E= E17.8,27HMAXIMUM DELTA E PERTURBED = |                  | 5833H144                   |
| 1 F17.8 )  |                  | *473                       |
| C32.0 RUN IS COMPLETED. CALL THE NEXT CASE AT STEP 4.2                 |                  | *474                       |
| GO TO 10   |                  | *475                       |
| END  |                  | 5835H144                   |
|  |                  | 5836H144                   |
|  |                  | *476                       |
|  |                  | 5838H144                   |
|  |                  | *477                       |

## Section 7

### SUBROUTINE FOR NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS, RKSIMP

#### 7.1 EQUATIONS IN ORDER OF SOLUTION

##### 7.1.1 Subroutine Input

###### 7.1.1.1 Calling Sequence

|            |   |
|------------|---|
| KR         | Runge-Kutta flag; takes on five values from 1 to 5 indicating the cycle of Runge-Kutta. Initially set = 1 by the main program and thereafter under the control of the subroutine. |
| IP         | Initial point flag; signals first point calculations by the subroutine. Initially set to 1 by the main program and advanced to 2 by the subroutine.                               |
| KC         | Simpson's rule computation flag. The computations are bypassed if KC = 1, computed if KC = 2. Wholly under the control of the subroutine.   |
| t          | Time; initial value set by main program, thereafter under control of the subroutine.  |
| $\Delta t$ | Computing interval; initial value is input; if input is left blank, it is set equal to 0.005 by the subroutine.   |
| ICH        | A flag that signals which equations are being integrated.<br>ICH = 1 corresponds to numerical residual perturbation solution  |

ICH = 2 corresponds to numerical solution of total equations of motion

Wholly under control of the main program.

$x_h, y_h, z_h$  Dummy variables of position, velocity, and acceleration, respectively.

$\dot{x}_h, \dot{y}_h, \dot{z}_h$

.. .. ..

$x_h, y_h, z_h$

KHALT Halt = 2, call next case.

$t_p$  Print time

$T_f$  Run stop time

#### 7.1.1.2 Common

W The input array

IPRINT Print flag; print if equal to 1.

Initially set = 1 by main program for numerical residual perturbation solution

Initially set = 2 by main program for numerical solution of total equations of motion

Under control of subroutine during numerical solution of total equations of motion

$t_s$  Time modulo the period of the precessing ellipse

$t_\psi$  Time since last passage of the apocenter by the analytic solution

$t_\Omega$  Time modulo the period of the node

$t_w$  Time modulo the period of the apocenter

$\gamma = \epsilon t$

$\epsilon \mu^{1/4}$

ITP Plot point index

#### 7.1.2 Test Runge-Kutta Flag, KR

If KR = 1, continue below (Step 7.1.3).

If KR = 2, go to Step 7.1.26.

If KR = 3, go to Step 7.1.27.

If KR = 4, go to Step 7.1.28.

If KR = 5, go to Step 7.1.30.

#### 7.1.3 Test Initial Flag, IP

If IP = 1, first point computations must be executed; continue below (Step 7.1.4).

If IP = 2, go to Step 7.1.9.

#### 7.1.4 First Point Computations

##### 7.1.4.1 Set Flags and Initial Conditions

Set IP = 2 Advance initial point flag.

LL = 0              Obsolete.

KT = -2              Set computing interval counter = -2.

ITP = 1              Set initial value of index for plotting arrays  $A_p$  and  $P_t$ .

KC = 1              Indicates next two Runge-Kutta cycles to be at the same  $\Delta_t$ .

$E_{\min} = |w_{12}|$  Set value for the minimum allowable error equal to input value.

FDT =  $|w_7|$  Modifier of allowable computing interval.

KF = 0              Set intermediate and total failure

KFAIL = 0              counters to zero.

MFAIL =  $|w_{11}|$  Set the maximum number of failures allowed equal to an input value.

$t_{pT} = P_{t1}$  Set first print time for numerical solution of total equations of motion.

(Note: Unnecessary on numerical residual perturbation solution, but saves coding if included here.)

|                          |   |
|--------------------------|---|
| $\Delta \dot{X} = 0$     | Sets velocity increments for use in computing interval tests initially equal to zero. |
| $\Delta \dot{Y} = 0$     |   |
| $\Delta \dot{Z} = 0$     |   |
| $\gamma = 0$             | Sets initial values of angle and time variables to zero.                              |
| $t_\psi = 0$             |   |
| $\beta = 0$              |   |
| $t_\Omega = 0$           |   |
| $t_\omega = 0$           |   |
| $t_s = 0$                |   |
| $\Delta t_p = a^{3/2}/4$ | Sets print interval to $1/4$ radian of M  |
| $t_p = \Delta t_p$       |   |

7.1.4.2 Set the Maximum Allowable Error Equal to the Input Value or Equal to  $10^{-7}$  if the Input Value is Zero

$$\epsilon_{all} = |w_8| \text{ or } \epsilon_{all} = 10^{-7} \text{ if } w_8 = 0$$

7.1.4.3 Set the Computing Interval Multiplier (Used if Error is Less Than the Minimum Allowed) Equal to Input Value or Equal to 1.5 if Input Value  $\leq 1$

$$\Delta t_{min} = w_9 \text{ or } \Delta t_{min} = 1.5 \text{ if } w_9 \leq 1$$

7.1.4.4 Set the Initial Computing Interval Equal to Input Value or Equal to 0.005 if the Input Value is Zero

$$\Delta t = w_{10} \text{ or } \Delta t = 0.005 \text{ if } w_{10} = 0$$

7.1.4.5 Test Perturbation - Total Flag

If ICH = 1, go to Step 7.1.5 (numerical residual perturbation solution).

If ICH = 2, go to Step 7.1.4.6 (numerical solution of total equations of motion).

7.1.4.6 Test the First Print Time in Numerical Solution of Total Equations of Motion Versus Twice the Computing Interval

If  $t_{pT} < 2\Delta t$ , go to Step 7.1.4.7.

If  $t_{pT} = 2\Delta t$ , go to Step 7.1.4.8.

If  $t_{pT} > 2\Delta t$ , go to Step 7.1.5.

7.1.4.7 Set  $\Delta t = \frac{1}{2} t_{pT}$

7.1.4.8 Set IPRINT = 3 to Signal Both Print and Testing of Computed Values

7.1.5 Save Quantities at Start of Span for Computing Interval Calculations

Save positions, velocities, accelerations, time,  $t_s$ ,  $t_\psi$ ,  $t_\Omega$ ,  $t_w$ ,  $\theta_1$ , and  $\sin \theta_1$  for restart in case computing interval selection fails and for use in Simpson's rule calculations.

The array SS(17) is used for this purpose with:

$ss_1 = \text{time}, t$

$ss_2, ss_3, ss_4 = \text{position coordinates}, x_h, y_h, z_h$

$ss_5, ss_6, ss_7 = \text{velocity coordinates}, \dot{x}_h, \dot{y}_h, \dot{z}_h$

$ss_8, ss_9, ss_{10} = \text{acceleration components}, \ddot{x}_h, \ddot{y}_h, \ddot{z}_h$

$ss_{11} = t_s$

$ss_{12} = t_\psi$

$ss_{13} = t_\Omega$

$ss_{14} = t_w$

$ss_{15} = \theta_1$

$ss_{16} = \sin \theta_1$

$ss_{17} = \gamma$

#### 7.1.5.1 Increment Computing Interval Counter

$$KT = KT + 2$$

#### 7.1.6 Save Positions, Velocities, Accelerations, and Time for Ordinary Runge-Kutta Use

The array S(17) is used for this purpose and the indices are ordered as is the SS array in Step 7.1.5.

#### 7.1.7 Compute the Next Value of Time (i.e., At the End of the Next Computing Interval) and Make Special Time Tests

$$TN = s_1 + \Delta t$$

##### 7.1.7.1 Test Perturbation - Total Flag

If ICH = 1 (numerical residual perturbation solutions), go to Step 7.1.7.3.

If ICH = 2 (numerical solutions of total equations of motion), continue below (Step 7.1.7.2).

##### 7.1.7.2 Test Run Stop Time Against the Next Print Time

If  $T_f > t_p$ , go to Step 7.1.7.6.

If  $T_f \leq t_p$ , continue below (Step 7.1.7.3).

##### 7.1.7.3 Test the Next Value of Time Against Run Stop Time

If  $TN > T_f$ , continue below (Step 7.1.7.4).

If  $TN = T_f$ , go to Step 7.1.7.5.

If  $TN < T_f$ , go to Step 7.1.8.

##### 7.1.7.4 Set Computing Interval So That Time Will Equal the Run Stop Time at the End of the Next Interval

$$\Delta t = T_f - t$$

#### 7.1.7.5 Set Print and Halt Flags

IPRINT = 1

KHALT = 3

Go to Step 7.1.8.

#### 7.1.7.6 Test the Next Value of Time Against the Next Print Time

(Enter here from Step 7.1.7.2 if  $T_f > t_p$ .)

If  $T_N > t_p$ , continue below (Step 7.1.7.7).

If  $T_N = t_p$ , go to Step 7.1.7.8.

If  $T_N < t_p$ , go to Step 7.1.8.

#### 7.1.7.7 Save Initial Conditions and Compute Interval for Print Only

Save the value of computing interval for use after print, and then set the computing interval so that the "next time" will equal the print time.

$$\Delta t_{sp} = \Delta t$$

$$\Delta t = t_p - t$$

Set the print flag.

IPRINT = 1

Save the values of positions, velocities, accelerations, and time in the array SP(10) for use after print. The indices correspond to those in Steps 7.1.5 and 7.1.6.

Set  $sp_I = s_I$  for  $I = 1, 2, \dots, 10$ .

Go to Step 7.1.8.

#### 7.1.7.8 Set IPRINT = 3

### 7.1.8 Complete First Runge-Kutta Pass

Compute intermediate value of time.

$$t = s_1 + \frac{1}{2} \Delta t$$

$$t_{\psi} = t_{\psi} + \frac{1}{2} \Delta t$$

$$t_s = t_s + \frac{1}{2} \Delta t$$

$$t_{\Omega} = t_{\Omega} + \frac{1}{2} \Delta t$$

$$t_{\omega} = t_{\omega} + \frac{1}{2} \Delta t$$

$$\gamma = \gamma + \frac{\epsilon \Delta t}{2}$$

Compute the intermediate position values.

$$x_h = s_2 + \frac{1}{2} \Delta t \times s_5$$

$$y_h = s_3 + \frac{1}{2} \Delta t \times s_6 \quad \text{parameters for (141)}$$

$$z_h = s_4 + \frac{1}{2} \Delta t \times s_7$$

Compute the Runge-Kutta parameters.

$$RK1X = \Delta t \times s_8$$

$$RK1Y = \Delta t \times s_9 \text{ computing intervals, etc. (140)}$$

$$RK1Z = \Delta t \times s_{10}$$

Set Runge-Kutta flag, KR = 2 and return to main program.

7.1.9 Test Whether Residual Perturbation of Total Equations are Being Solved

(Enter here from Step 7.1.3 on a first pass of Runge-Kutta which is not the first point of the trajectory.)

If ICH = 1 (residual perturbation), go to Step 7.1.12.

If ICH = 2 (total), continue below (Step 7.1.10).

7.1.10 Test Print Flag

If IPRINT = 1 (print), continue below (Step 7.1.11).

If IPRINT = 2 or 3 (don't print), go to Step 7.1.12.

7.1.11 Advance Print Index and Set the Next Print Time

$$ITP = ITP + 1$$

$$t_{PT} = P_t(ITP)$$

If IPRINT = 1, go to Step 7.1.11.1.

If IPRINT = 2, go to Step 7.1.5.

If IPRINT = 3, go to Step 7.1.11.2.

7.1.11.1 Restore Values Saved Previously in Step 7.1.7.7

$$\Delta t = \Delta t_{sp}$$

$$s_I = s_{I-1} \text{ for } I = 1, 2, \dots, 10$$

Set IPRINT = 2

Return to Step 7.1.7.

7.1.11.2 Set IPRINT = 2 and Go to Step 7.1.5

#### 7.1.12 Test Simpson's Rule Flag

(Enter here from Step 7.1.9 when numerical residual perturbation solution is being computed and from Step 7.1.10 on a nonprint point during the numerical solution of the total equations of motion.)

If KC = 1, continued below (Step 7.1.13).

If KC = 2, go to Step 7.1.14.

#### 7.1.13 Advance Simpson's Rule Flag

KC = 1 signals that the first phase of the two-cycle Runge-Kutta is in progress. KC = 2, and return to Step 7.1.6.

#### 7.1.14 Reset Simpson's Rule Flag and Compute Actual and Allowable Errors

(Enter here from Step 7.1.12 if KC = 2. Signals that two Runge-Kutta cycles have been completed and accuracy tests can now be made.)

KC = 1

Compute the velocity increments by Simpson's rule:

$$\dot{x}_h = 1/3 \Delta t (ss_8 + 4s_8 + \ddot{x}_h)$$

$$\dot{y}_h = 1/3 \Delta t (ss_9 + 4s_9 + \ddot{y}_h)$$

$$\dot{z}_h = 1/3 \Delta t (ss_{10} + 4s_{10} + \ddot{z}_h)$$

Compute the estimated error (E) as follows:

Set  $C_{\max}$  = the maximum absolute value of the Runge-Kutta velocity increments over the last two cycles of Runge-Kutta  $\Delta X$ ,  $\Delta Y$ ,  $\Delta Z$ .

Determine E as the maximum absolute value of the respective differences between the Simpson's rule and Runge-Kutta velocity increments.

Reset the Runge-Kutta increments to zero for proper computation during the next two cycles:

$$\Delta \dot{X} = \Delta \dot{Y} = \Delta \dot{Z} = 0$$

Compute maximum and minimum allowable errors:

$$E_{\text{all}} = \max (|w_8|c_{\text{max}}, 10^{-9}\text{max}, (|\dot{x}_h|, |\dot{y}_h|, |\dot{z}_h|))$$

$$E_{\text{min}} = w_{12} c_{\text{max}}$$

#### 7.1.15 Test the Estimated Error Against the Maximum Allowable Error

If  $E > E_{\text{all}}$ , continue below (Step 7.1.16).

If  $E < E_{\text{all}}$ , go to Step 7.1.22.

#### 7.1.16 Increment and Test the Total Failure Counter

$$\text{KFAIL} = \text{KFAIL} + 1$$

Test the number of failures against the maximum allowed.

If  $\text{KFAIL} > \text{MFAIL}$ , continue below (Step 7.1.17).

If  $\text{KFAIL} < \text{MFAIL}$ , go to Step 7.1.18.

#### 7.1.17 Set Halt Flag to Stop Trajectory Computations and Compute Final Time

$$\text{KHALT} = 3$$

$$T_f = t$$

Go to step 7.1.29.

7.1.18 Increment the Intermediate Failure Counter

(Enter here from Step 7.1.16 if the number of failures is less than the maximum allowed.)

KF = KF + 1

KHALT = 1

7.1.19 Compute a New Δt Based on the Value of Estimated Error, E

$$\Delta t = FDT \cdot \Delta t \left( E_{all}/E \right)^{1/4}$$

If  $\Delta t/t > 10^{-8}$ , go to Step 7.1.20; otherwise print the computer interval and go to Step 7.1.17.

7.1.20 Test Intermediate Failure Counter

If KF  $\leq 0$ , continue below (Step 7.1.20.1).

If KF  $> 0$ , go to Step 7.1.21.

7.1.20.1 Test Perturbation - Total Flag

If ICH = 1, go to Step 7.1.20.2.

If ICH = 2, go to Step 7.1.20.3.

7.1.20.2 Perturbation Solution Calculations

KR = 5

7.1.20.2.1 Check time against print time

If  $t \geq t_p$ , go to Step 7.1.20.2.2.

If  $t < t_p$ , go to Step 7.1.5.

7.1.20.2.2 Increment print time

$$t_p = t_p + \Delta t_p$$

Go to Step 7.1.29.

7.1.20.3 Test if This is Both a Print and a Regular Compute Point (IPRINT=3)

If IPRINT = 1 or 2, go to Step 7.1.5.

If IPRINT = 3, go to Step 7.1.20.2.

7.1.21 Restore Values Saved in Step 7.1.5 to Those in Step 7.1.6

Set  $s_I = ss_I$  for  $I = 1, 2, \dots, 17$

IPRINT = 2

Return to Step 7.1.8.

7.1.22 Test the Estimated Error Against the Minimum Allowable Error,  $E_{min}$

(Enter from Step 7.1.15 if the estimated error < the maximum allowed.)

If  $E \leq E_{min}$ , continue below (Step 7.1.23).

If  $E > E_{min}$ , go to Step 7.1.25.

7.1.23 Increment Total Failure Counter

$$KFAIL = KFAIL + 1$$

Test the number of failures against the maximum allowed.

If KFAIL > MFAIL, return to Step 7.1.17.

If KFAIL  $\leq$  MFAIL, continued below (Step 7.1.24).

7.1.24 Increment Intermediate Failure Counter, Compute New  $\Delta t$ , and Restore Initial Conditions

KF = KF + 1

KHALT = 1

Compute new  $\Delta t$  by multiplying the old by an input factor (>1):

$$\Delta t = \Delta t_{\min} \Delta t$$

Restore the values saved in Step 7.1.5 to those in Step 7.1.6.

Set  $s_I = ss_I$  for  $I = 1, 2, \dots, 17$

Return to Step 7.1.17.

7.1.25 Set the Intermediate Failure Counter to Zero

(Enter here from Step 7.1.22 with  $E_{\min} < E < E_{\text{all}}$ .)

KF = 0

7.1.26 Second Pass Computations

(Enter here on second pass of Runge-Kutta (KR = 2).)

Advance Runge-Kutta:

KR = 3

Compute new intermediate position values:

$$x_h = x_h + \frac{1}{4} (\Delta t) \quad (\text{RKIX})$$

$$y_h = y_h + \frac{1}{4} (\Delta t) \quad (\text{RKIY}) \quad \text{parameters for (142)}$$

$$z_h = z_h + \frac{1}{4} (\Delta t) \quad (\text{RKIZ})$$

Compute Runge-Kutta parameters:

$$RK2X = \Delta t \ddot{x}_h$$

$$RK2Y = \Delta t \ddot{y}_h \quad (141)$$

$$RK2Z = \Delta t \ddot{z}_h$$

Return to main program.

#### 7.1.27 Third Pass Computations

(Enter here on third pass of Runge-Kutta (KR = 3).)

Advance Runge-Kutta flag:

$$KR = 4$$

Compute new time and intermediate position values:

$$t_\psi = t_\psi + \frac{1}{2} \Delta t$$

$$t_s = t_s + \frac{1}{2} \Delta t$$

$$t_\Omega = t_\Omega + \frac{1}{2} \Delta t$$

$$t_\omega = t_\omega + \frac{1}{2} \Delta t$$

$$\gamma = \gamma + \epsilon \Delta t / 2$$

$$T = s_1 + \Delta t$$

Compute Runge-Kutta parameters:

$$x_h = s_2 + \Delta t \times s_5 + \frac{1}{2} \Delta t \times RK2X$$

$$y_h = s_3 + \Delta t \times s_6 + \frac{1}{2} \Delta t \times RK2Y \quad \text{parameters for (143)}$$

$$z_h = s_4 + \Delta t \times s_7 + \frac{1}{2} \Delta t \times RK2Z$$

$$RK3X = \Delta t \ddot{x}_h$$

$$RK3Y = \Delta t \ddot{y}_h \quad (142)$$

$$RK3Z = \Delta t \ddot{z}_h$$

Return to main program.

### 7.1.28 Fourth Pass Computations

(Enter here on fourth pass of Runge-Kutta (KR = 4).)

Reset Runge-Kutta flag:

$$KR = 1$$

Compute integrated position, velocities, and velocity increments.

$$x_h = s_2 + \Delta t \times s_5 + \frac{1}{6} \Delta t \quad (RK1X + RK2X + RK3X)$$

$$y_h = s_3 + \Delta t \times s_6 + \frac{1}{6} \Delta t \quad (RK1Y + RK2Y + RK3Y) \quad (144)$$

$$z_h = s_4 + \Delta t \times s_7 + \frac{1}{6} \Delta t \quad (RK1Z + RK2Z + RK3Z)$$

$$\dot{x}_h = s_5 + \frac{1}{6} (RK1X + 2(RK2X + RK3X) + \Delta t \ddot{x}_h) \quad (143,$$

$$\dot{y}_h = s_6 + \frac{1}{6} (RK1Y + 2(RK2Y + RK3Y) + \Delta t \ddot{y}_h) \quad (145)$$

$$\dot{z}_h = s_7 + \frac{1}{6} (RK1Z + 2(RK2Z + RK3Z) + \Delta t \ddot{z}_h)$$

7.1.28.1 Test Whether Residual Perturbation Equations or Total Equations of Motion are Being Integrated

If ICH = 1 (perturbation), go to Step 7.1.28.3.

If ICH = 2 (total), go to Step 7.1.28.2.

7.1.28.2 Test Value of Print Flag

If IPRINT = 1, go to Step 7.1.29.

If IPRINT = 2 or 3, go to Step 7.1.28.3.

7.1.28.3 Compute Two Interval Increments in Velocity

$$\Delta \dot{X} = \dot{X} + \dot{x}_h - s_5$$

$$\Delta \dot{Y} = \dot{Y} + \dot{y}_h - s_6$$

$$\Delta \dot{Z} = \dot{Z} + \dot{z}_h - s_7$$

7.1.29 Return to Main Program

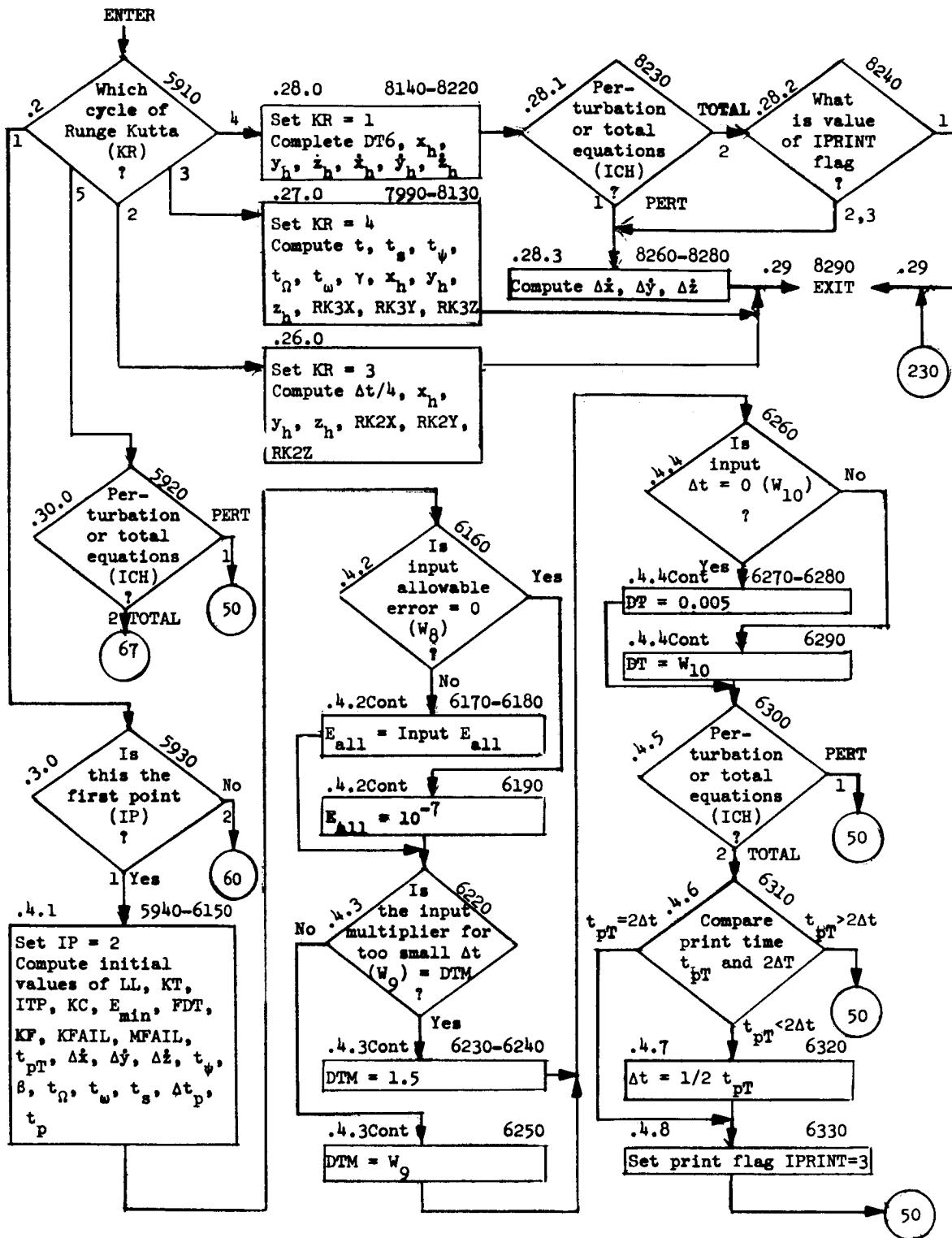
7.1.30 Test Whether Residual Perturbation or Total Equations of Motion are Being Integrated

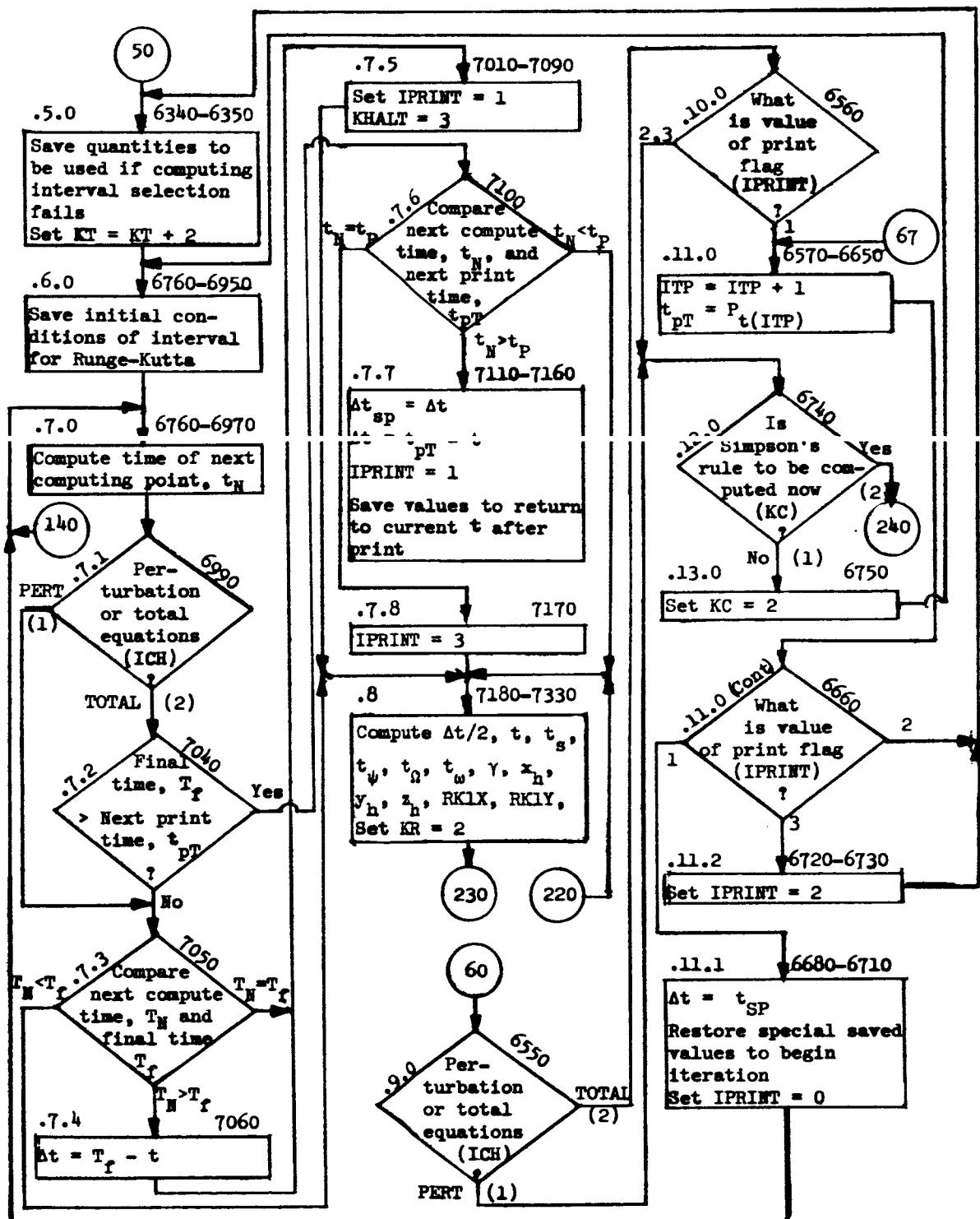
If ICH = 1 (perturbation), go to Step 7.1.5.

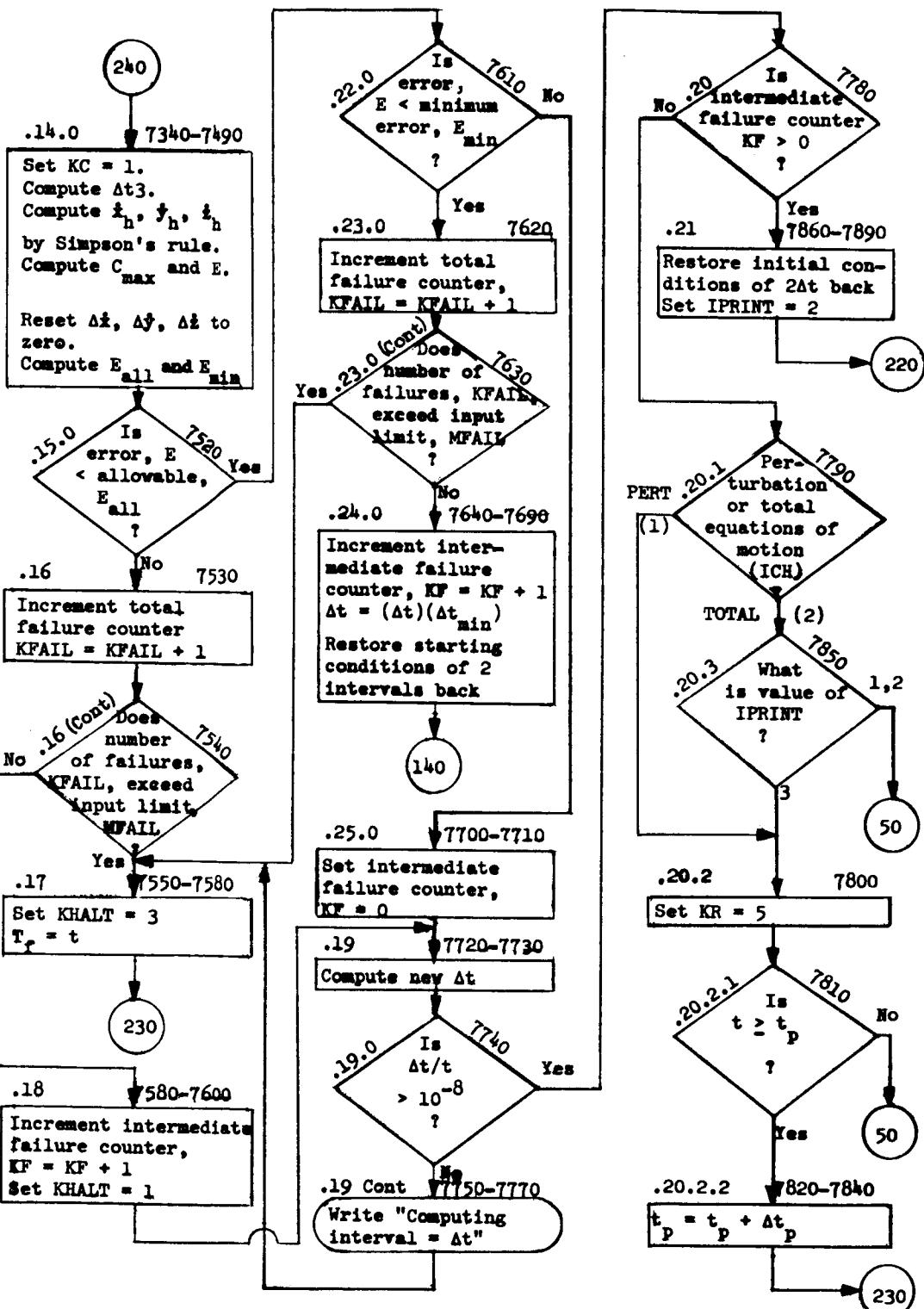
If ICH = 2 (total), go to Step 7.1.11.

## 7.2 SUBROUTINE RKSIMP DETAIL FLOW CHARTS

The flow charts of this section describe the logic of subroutine RKSIMP completely, i.e., all conditional transfers are shown. The numbers to the top left of each box are the subsection numbers of Section 7.1 where the operations mentioned in the box are detailed. The numbers at the top right are the card numbers of the subroutine listing, Section 7.3.







### 7.3 RKSIMP LISTING

This section contains the listing for subroutine RKSIMP. The subsection numbers on the comment cards refer to the subsections in Section 7.1, wherein the code is explained.

K144 RKSIMP FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S) 09/01/65

```

SUBROUTINE RKSSIMP (KR, IP, KC, T, DT, ICH, HX, HY, HZ,
1 HXD, HYD, HZD, HY2D, HZ2D, KHALJ, TPINT, TF)
2 COMMN W, TPINT, TS, TPSI, TNOD, TPER, GAMMA, EPI,
3 LTP, PTIME, AP
4 DIMENSION W(30), S(17), SS(17), SP(17),
5 PTIMEL(7001),
6 PTST RUNGF-KUTTA FLAG, KR
7 GO TO 110, 360, 370, 380, 51, KR
8 TEST WHETHER RESIDUAL PERTURBATION OR TOTAL EQUATIONS OF MOTION
9 5850H144
10 5910H144
11 5915H144
12 5916H144 *1
13 5920H144
14 5925H144 *2
15 5930H144
16 5933H144
17 5937H144 *3
18 5940H144
19 5945H144 *4
20 5950H144
21 5960H144 *5
22 5970H144
23 5980H144 *6
24 5990H144
25 6000H144 *7
26 6010H144
27 6020H144
28 6030H144
29 6040H144
30 6050H144
31 6060H144
32 6070H144
33 6080H144
34 6090H144
35 6100H144
36 6110H144
37 6120H144
38 6130H144
39 6140H144
40 6150H144
41 6160H144
42 6170H144
43 6180H144
44 6190H144
45 6215H144
46 6216H144
47 6217H144
48 6220H144
49 6226H144
50 6229H144
51 6240H144
52 6250H144
53 6255H144
54 6256H144
55 6260H144
56 6270H144
57 6280H144
58 6296H144
59 6301H44
60 6305H144
61 6310H144
62 6315H144
63 6320H144
64 6325H144
65 6330H144
66 6335H144
67 6340H144
68 6345H144
69 6350H144
70 6355H144
71 6360H144
72 6365H144
73 6370H144
74 6375H144
75 6380H144
76 6385H144
77 6390H144
78 6395H144
79 6400H144
80 6405H144
81 6410H144
82 6415H144
83 6420H144
84 6425H144
85 6430H144
86 6435H144
87 6440H144
88 6445H144
89 6450H144
90 6455H144
91 6460H144
92 6465H144
93 6470H144
94 6475H144
95 6480H144
96 6485H144
97 6490H144
98 6495H144
99 6500H144
100 6505H144
101 6510H144
102 6515H144
103 6520H144
104 6525H144
105 6530H144
106 6535H144
107 6540H144
108 6545H144
109 6550H144
110 6555H144
111 6560H144
112 6565H144
113 6570H144
114 6575H144
115 6580H144
116 6585H144
117 6590H144
118 6595H144
119 6600H144
120 6605H144
121 6610H144
122 6615H144
123 6620H144
124 6625H144
125 6630H144
126 6635H144
127 6640H144
128 6645H144
129 6650H144
130 6655H144
131 6660H144
132 6665H144
133 6670H144
134 6675H144
135 6680H144
136 6685H144
137 6690H144
138 6695H144
139 6700H144
140 6705H144
141 6710H144
142 6715H144
143 6720H144
144 6725H144
145 6730H144
146 6735H144
147 6740H144
148 6745H144
149 6750H144
150 6755H144
151 6760H144
152 6765H144
153 6770H144
154 6775H144
155 6780H144
156 6785H144
157 6790H144
158 6795H144
159 6800H144
160 6805H144
161 6810H144
162 6815H144
163 6820H144
164 6825H144
165 6830H144
166 6835H144
167 6840H144
168 6845H144
169 6850H144
170 6855H144
171 6860H144
172 6865H144
173 6870H144
174 6875H144
175 6880H144
176 6885H144
177 6890H144
178 6895H144
179 6900H144
180 6905H144
181 6910H144
182 6915H144
183 6920H144
184 6925H144
185 6930H144
186 6935H144
187 6940H144
188 6945H144
189 6950H144
190 6955H144
191 6960H144
192 6965H144
193 6970H144
194 6975H144
195 6980H144
196 6985H144
197 6990H144
198 6995H144
199 7000H144
200 7005H144
201 7010H144
202 7015H144
203 7020H144
204 7025H144
205 7030H144
206 7035H144
207 7040H144
208 7045H144
209 7050H144
210 7055H144
211 7060H144
212 7065H144
213 7070H144
214 7075H144
215 7080H144
216 7085H144
217 7090H144
218 7095H144
219 7100H144
220 7105H144
221 7110H144
222 7115H144
223 7120H144
224 7125H144
225 7130H144
226 7135H144
227 7140H144
228 7145H144
229 7150H144
230 7155H144
231 7160H144
232 7165H144
233 7170H144
234 7175H144
235 7180H144
236 7185H144
237 7190H144
238 7195H144
239 7200H144
240 7205H144
241 7210H144
242 7215H144
243 7220H144
244 7225H144
245 7230H144
246 7235H144
247 7240H144
248 7245H144
249 7250H144
250 7255H144
251 7260H144
252 7265H144
253 7270H144
254 7275H144
255 7280H144
256 7285H144
257 7290H144
258 7295H144
259 7300H144
260 7305H144
261 7310H144
262 7315H144
263 7320H144
264 7325H144
265 7330H144
266 7335H144
267 7340H144
268 7345H144
269 7350H144
270 7355H144
271 7360H144
272 7365H144
273 7370H144
274 7375H144
275 7380H144
276 7385H144
277 7390H144
278 7395H144
279 7400H144
280 7405H144
281 7410H144
282 7415H144
283 7420H144
284 7425H144
285 7430H144
286 7435H144
287 7440H144
288 7445H144
289 7450H144
290 7455H144
291 7460H144
292 7465H144
293 7470H144
294 7475H144
295 7480H144
296 7485H144
297 7490H144
298 7495H144
299 7500H144
300 7505H144
301 7510H144
302 7515H144
303 7520H144
304 7525H144
305 7530H144
306 7535H144
307 7540H144
308 7545H144
309 7550H144
310 7555H144
311 7560H144
312 7565H144
313 7570H144
314 7575H144
315 7580H144
316 7585H144
317 7590H144
318 7595H144
319 7600H144
320 7605H144
321 7610H144
322 7615H144
323 7620H144
324 7625H144
325 7630H144
326 7635H144
327 7640H144
328 7645H144
329 7650H144
330 7655H144
331 7660H144
332 7665H144
333 7670H144
334 7675H144
335 7680H144
336 7685H144
337 7690H144
338 7695H144
339 7700H144
340 7705H144
341 7710H144
342 7715H144
343 7720H144
344 7725H144
345 7730H144
346 7735H144
347 7740H144
348 7745H144
349 7750H144
350 7755H144
351 7760H144
352 7765H144
353 7770H144
354 7775H144
355 7780H144
356 7785H144
357 7790H144
358 7795H144
359 7800H144
360 7805H144
361 7810H144
362 7815H144
363 7820H144
364 7825H144
365 7830H144
366 7835H144
367 7840H144
368 7845H144
369 7850H144
370 7855H144
371 7860H144
372 7865H144
373 7870H144
374 7875H144
375 7880H144
376 7885H144
377 7890H144
378 7895H144
379 7900H144
380 7905H144
381 7910H144
382 7915H144
383 7920H144
384 7925H144
385 7930H144
386 7935H144
387 7940H144
388 7945H144
389 7950H144
390 7955H144
391 7960H144
392 7965H144
393 7970H144
394 7975H144
395 7980H144
396 7985H144
397 7990H144
398 7995H144
399 8000H144
400 8005H144
401 8010H144
402 8015H144
403 8020H144
404 8025H144
405 8030H144
406 8035H144
407 8040H144
408 8045H144
409 8050H144
410 8055H144
411 8060H144
412 8065H144
413 8070H144
414 8075H144
415 8080H144
416 8085H144
417 8090H144
418 8095H144
419 8100H144
420 8105H144
421 8110H144
422 8115H144
423 8120H144
424 8125H144
425 8130H144
426 8135H144
427 8140H144
428 8145H144
429 8150H144
430 8155H144
431 8160H144
432 8165H144
433 8170H144
434 8175H144
435 8180H144
436 8185H144
437 8190H144
438 8195H144
439 8200H144
440 8205H144
441 8210H144
442 8215H144
443 8220H144
444 8225H144
445 8230H144
446 8235H144
447 8240H144
448 8245H144
449 8250H144
450 8255H144
451 8260H144
452 8265H144
453 8270H144
454 8275H144
455 8280H144
456 8285H144
457 8290H144
458 8295H144
459 8300H144
460 8305H144
461 8310H144
462 8315H144
463 8320H144
464 8325H144
465 8330H144
466 8335H144
467 8340H144
468 8345H144
469 8350H144
470 8355H144
471 8360H144
472 8365H144
473 8370H144
474 8375H144
475 8380H144
476 8385H144
477 8390H144
478 8395H144
479 8400H144
480 8405H144
481 8410H144
482 8415H144
483 8420H144
484 8425H144
485 8430H144
486 8435H144
487 8440H144
488 8445H144
489 8450H144
490 8455H144
491 8460H144
492 8465H144
493 8470H144
494 8475H144
495 8480H144
496 8485H144
497 8490H144
498 8495H144
499 8500H144
500 8505H144
501 8510H144
502 8515H144
503 8520H144
504 8525H144
505 8530H144
506 8535H144
507 8540H144
508 8545H144
509 8550H144
510 8555H144
511 8560H144
512 8565H144
513 8570H144
514 8575H144
515 8580H144
516 8585H144
517 8590H144
518 8595H144
519 8600H144
520 8605H144
521 8610H144
522 8615H144
523 8620H144
524 8625H144
525 8630H144
526 8635H144
527 8640H144
528 8645H144
529 8650H144
530 8655H144
531 8660H144
532 8665H144
533 8670H144
534 8675H144
535 8680H144
536 8685H144
537 8690H144
538 8695H144
539 8700H144
540 8705H144
541 8710H144
542 8715H144
543 8720H144
544 8725H144
545 8730H144
546 8735H144
547 8740H144
548 8745H144
549 8750H144
550 8755H144
551 8760H144
552 8765H144
553 8770H144
554 8775H144
555 8780H144
556 8785H144
557 8790H144
558 8795H144
559 8800H144
560 8805H144
561 8810H144
562 8815H144
563 8820H144
564 8825H144
565 8830H144
566 8835H144
567 8840H144
568 8845H144
569 8850H144
570 8855H144
571 8860H144
572 8865H144
573 8870H144
574 8875H144
575 8880H144
576 8885H144
577 8890H144
578 8895H144
579 8900H144
580 8905H144
581 8910H144
582 8915H144
583 8920H144
584 8925H144
585 8930H144
586 8935H144
587 8940H144
588 8945H144
589 8950H144
590 8955H144
591 8960H144
592 8965H144
593 8970H144
594 8975H144
595 8980H144
596 8985H144
597 8990H144
598 8995H144
599 9000H144
600 9005H144
601 9010H144
602 9015H144
603 9020H144
604 9025H144
605 9030H144
606 9035H144
607 9040H144
608 9045H144
609 9050H144
610 9055H144
611 9060H144
612 9065H144
613 9070H144
614 9075H144
615 9080H144
616 9085H144
617 9090H144
618 9095H144
619 9100H144
620 9105H144
621 9110H144
622 9115H144
623 9120H144
624 9125H144
625 9130H144
626 9135H144
627 9140H144
628 9145H144
629 9150H144
630 9155H144
631 9160H144
632 9165H144
633 9170H144
634 9175H144
635 9180H144
636 9185H144
637 9190H144
638 9195H144
639 9200H144
640 9205H144
641 9210H144
642 9215H144
643 9220H144
644 9225H144
645 9230H144
646 9235H144
647 9240H144
648 9245H144
649 9250H144
650 9255H144
651 9260H144
652 9265H144
653 9270H144
654 9275H144
655 9280H144
656 9285H144
657 9290H144
658 9295H144
659 9300H144
660 9305H144
661 9310H144
662 9315H144
663 9320H144
664 9325H144
665 9330H144
666 9335H144
667 9340H144
668 9345H144
669 9350H144
670 9355H144
671 9360H144
672 9365H144
673 9370H144
674 9375H144
675 9380H144
676 9385H144
677 9390H144
678 9395H144
679 9400H144
680 9405H144
681 9410H144
682 9415H144
683 9420H144
684 9425H144
685 9430H144
686 9435H144
687 9440H144
688 9445H144
689 9450H144
690 9455H144
691 9460H144
692 9465H144
693 9470H144
694 9475H144
695 9480H144
696 9485H144
697 9490H144
698 9495H144
699 9500H144
700 9505H144
701 9510H144
702 9515H144
703 9520H144
704 9525H144
705 9530H144
706 9535H144
707 9540H144
708 9545H144
709 9550H144
710 9555H144
711 9560H144
712 9565H144
713 9570H144
714 9575H144
715 9580H144
716 9585H144
717 9590H144
718 9595H144
719 9600H144
720 9605H144
721 9610H144
722 9615H144
723 9620H144
724 9625H144
725 9630H144
726 9635H144
727 9640H144
728 9645H144
729 9650H144
730 9655H144
731 9660H144
732 9665H144
733 9670H144
734 9675H144
735 9680H144
736 9685H144
737 9690H144
738 9695H144
739 9700H144
740 9705H144
741 9710H144
742 9715H144
743 9720H144
744 9725H144
745 9730H144
746 9735H144
747 9740H144
748 9745H144
749 9750H144
750 9755H144
751 9760H144
752 9765H144
753 9770H144
754 9775H144
755 9780H144
756 9785H144
757 9790H144
758 9795H144
759 9800H144
760 9805H144
761 9810H144
762 9815H144
763 9820H144
764 9825H144
765 9830H144
766 9835H144
767 9840H144
768 9845H144
769 9850H144
770 9855H144
771 9860H144
772 9865H144
773 9870H144
774 9875H144
775 9880H144
776 9885H144
777 9890H144
778 9895H144
779 9900H144
780 9905H144
781 9910H144
782 9915H144
783 9920H144
784 9925H144
785 9930H144
786 9935H144
787 9940H144
788 9945H144
789 9950H144
790 9955H144
791 9960H144
792 9965H144
793 9970H144
794 9975H144
795 9980H144
796 9985H144
797 9990H144
798 9995H144
799 9999H144
800 9999H144
801 9999H144
802 9999H144
803 9999H144
804 9999H144
805 9999H144
806 9999H144
807 9999H144
808 9999H144
809 9999H144
810 9999H144
811 9999H144
812 9999H144
813 9999H144
814 9999H144
815 9999H144
816 9999H144
817 9999H144
818 9999H144
819 9999H144
820 9999H144
821 9999H144
822 9999H144
823 9999H144
824 9999H144
825 9999H144
826 9999H144
827 9999H144
828 9999H144
829 9999H144
830 9999H144
831 9999H144
832 9999H144
833 9999H144
834 9999H144
835 9999H144
836 9999H144
837 9999H144
838 9999H144
839 9999H144
840 9999H144
841 9999H144
842 9999H144
843 9999H144
844 9999H144
845 9999H144
846 9999H144
847 9999H144
848 9999H144
849 9999H144
850 9999H144
851 9999H144
852 9999H144
853 9999H144
854 9999H144
855 9999H144
856 9999H144
857 9999H144
858 9999H144
859 9999H144
860 9999H144
861 9999H144
862 9999H144
863 9999H144
864 9999H144
865 9999H144
866 9999H144
867 9999H144
868 9999H144
869 9999H144
870 9999H144
871 9999H144
872 9999H144
873 9999H144
874 9999H144
875 9999H144
876 9999H144
877 9999H144
878 9999H144
879 9999H144
880 9999H144
881 9999H144
882 9999H144
883 9999H144
884 9999H144
885 9999H144
886 9999H144
887 9999H144
888 9999H144
889 9999H144
890 9999H144
891 9999H144
892 9999H144
893 9999H144
894 9999H144
895 9999H144
896 9999H144
897 9999H144
898 9999H144
899 9999H144
900 9999H144
901 9999H144
902 9999H144
903 9999H144
904 9999H144
905 9999H144
906 9999H144
9
```

P144 RKSIMP EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INITIAL FORMULA NUMBER(S) 09/01/65  
 47 DT = W10  
 C4-5 TFST PERTURBATION-TOTAL FLAG  
 GO TO 150. 43) ICH  
 C4-6 TFST FIRST PRINT TIME IN NUMERICAL SOLUTION OF TOTAL EQUATIONS JF 6300H144  
 C MOTION VS. TWICE THE COMPUTING INTERVAL  
 C 43 IF (IPRINT - 2). \* DT) 44. 46. 50  
 C4-7 SFT DT = 1/2 T SUB PT  
 C4-8 CT = 0.5 \* SPRINT  
 C4-9 SFT IPRINT = 3 TO SIGNAL BOTH PRINT AND TESTING OF COMPUTED VALUES 6320H144  
 C4-10 IPRINT = 3  
 C5-0 SAVF QUANTITIES AT START OF SPAN FOR COMPUTING INTERVAL  
 C CALCULATIONS  
 50 SS(11) = T  
 SS(12) = MX  
 SS(13) = HY  
 SS(14) = HZ  
 SS(15) = HXO  
 SS(16) = HYD  
 SS(17) = HZD  
 SS(18) = HX2D  
 SS(19) = HY2D  
 SS(10) = HZ2D  
 SS(11) = TS  
 SS(12) = TPSI  
 SS(13) = TNCD  
 SS(14) = TPER  
 SS(15) = TMFL1  
 SS(16) = STHFL1  
 SS(17) = GAMMA  
 C9.1 INCREMENT COMPUTING INTERVAL COUNTER  
 KT = KT + 2  
 GO TO 110  
 C9.0 TEST WHETHER RESIDUAL PERTURBATION OR TOTAL EQUATIONS ARE BEING  
 C SOLVED  
 C 60 GO TO 190. 63) ICH  
 C10-0 TFST THE PRINT FLAG  
 C10-1 GO TO (167. 90. 90). IPRINT  
 C11-0 ADVANCE PRINT INDEX AND SET NEXT PRINT TIME  
 C11-1 ITP = ITP + 1  
 C11-2 IPRINT = PTIME (ITP)  
 C11-3 IF (IPRINT - 2). 75. 50. 85  
 C11-4 RESTORE VALUES SAVED PREVIOUSLY IN STEP 7.1.7.7  
 C11-5 DT = SPDT  
 C11-6 CO AC I = 1. 17  
 C11-7 GO TO 50  
 C11-8 IPRINT = 2  
 C11-9 GO TO 140  
 C11-10 SET IPRINT = 2 AND GO TO STEP 7.1.5  
 C11-11 IPRINT = 2  
 C12-0 TEST THE SIMPSON'S RULE FLAG  
 C12-1 GO TO 110. 240. KC  
 C13-0 ADVANCE THE SIMPSON'S RULE FLAG  
 C13-1 KC 2  
 C6-0 SAVE POSITIONS. VELOCITIES, ACCELERATIONS AND TIME FOR  
 C ORDINARY RUNGE-KUTTA USE  
 C770H144 .75

H144 RKSIMP EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S) 09/01/65  
 110 S(1) = T  
 S(2) = HX  
 S(3) = HY  
 S(4) = HZ  
 S(5) = HXD  
 S(6) = HYD  
 S(7) = HZD  
 S(8) = HXD0  
 S(9) = HYD0  
 S(10) = HZD0  
 S(11) = TS  
 S(12) = TPSI  
 S(13) = TNOD  
 S(14) = TPFR  
 S(15) = THFTAI  
 S(16) = STHFTI  
 S(17) = GAMMA  
 GO TO 140  
 C7.0 COMPUTE THE NEXT VALUE OF TIME AND MAKE SPECIAL PRINT TIME TESTS  
 140 TN = S(11) + DT  
 C7.1 TFST PERTURBATION-TOTAL FLAG  
 139 GO TO 1150. 1451. ICH  
 C7.2 TFST RUN STOP TIME AGAINST THE NEXT PRINT TIME  
 145 IF (TF - TPINT) 150. 150. 180  
 C7.3 TFST THE NEXT VALUE OF TIME AGAINST RUN STOP TIME  
 150 IF (TN - TF) 220. 170. 160  
 C7.4 SET COMPUTING INTERVAL SO THAT TIME WILL EQUAL THE RUN STOP TIME  
 C AT THE END OF THE NEXT INTERVAL  
 160 CI = TF - T  
 C7.5 SET THE PRINT AND HALT FLAGS  
 170 IPRINT = 1  
 KHALT = 3  
 GO TO 220  
 C7.6 TFST THE NEXT VALUE OF TIME AGAINST THE NEXT PRINT TIME  
 180 IF (TN - TPINT) 220. 200. 190  
 C7.7 SAVF INITIAL CONDITIONS AND COMPUTE INTERVAL FOR PRINT ONLY  
 190 SPDT = DT  
 DT = TPINT - T  
 IPRINT = 1  
 CO 210 I = 1. 17  
 210 SP(1) = S(1)  
 GO TO 220  
 C7.8 INDICATE COMPUTE POINT = PRINT POINT  
 200 IPRINT = 3  
 CA-0 COMPLETIF 1ST R-K PASS, COMPUTE NEW TIME AND POSITIONS  
 220 DT2 = DT / 2.  
 T = S(11) + DT2  
 TS = S(11) + DT2  
 TPSI = S(12) + DT2  
 TNOD = S(13) + DT2  
 TPFR = S(14) + DT2  
 GAMMA = S(17) + EP1 \* DT2  
 HX = S(2) + DT2 \* S(5)  
 HY = S(3) + DT2 \* S(6)  
 HZ = S(4) + DT2 \* S(7)  
 RK1X = DT \* S(8)

W144 RKSIMP            EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)            09/01/65

```

RKIV = DT * S(9)
RK17 = DT * S(10)
KR = ?
901 GO TO 230
C14.0 RESFT SIMPSONS RULE FLAG AND COMPUTE ACTUAL AND ALLOWABLE ERRORS
240 KC = 1
013 = DT / 3.
HMDA = DT3 * (SS(8) + 4. * S(8) + HX2D)
HYDA = DT3 * (SS(9) + 4. * S(9) + HY2D)
H2DA = DT3 * (SS(10) + 4. * S(10) + HZ2D)
C COMPUTE MAXIMUM ESTIMATED ERROR
CMAA = AMAX1 (ABS(DEXD)* ABS(DELYD)* ABS(DELU))
ESTFR = AMAX1 (ABS(DELU - HADA). ABS(DELYD - HYDA).
1 LRS (D(ZU - HADA))
DEXD = 0.
DELYD = 0.
DELU = 0.
FALI = AMAX1 (ABS (W(8)) * CMAX. 1.E-9 * AMAX1 (ABS (HXD
1.10 ABS (HYD). ABS (HZD)))
FMIN = ABS (W(12)) * CMAX
C15.0 TFST THE ESTIMATED ERROR AGAINST THE MAXIMUM ALLOWABLE ERROR
IF (ESTER - FALI) 280, 280, 250
C16.0 INCREMENT AND TEST THE TOTAL FAILURE COUNTER
250 KFAIL = KFAIL + 1
IF (KFAIL - MFAIL) 270, 270, 260
C17.0 SET HALT FLAG TO STOP TRAJECTORY COMPUTATIONS AND COMPUTE FINAL
C TIMEF
260 KHAI T = 3
TF = T
GO TO 230
C18.0 INCRFMNT THF INTERMEDIATE FAILURE COUNTER
270 KF = KF + 1
KHAI = 1
GO TO 330
C22.0 TEST THE ESTIMATED ERROR AGAINST THE MINIMUM ALLOWABLE ERROR
C (FRMIN)
280 IF (ESTER - ERMIN) 290, 290, 320
C23.0 INCREMENT THE TOTAL FAILURE COUNTER AND TEST THE NUMBER OF
C FAILURES AGAINST THE MAXIMUM ALLOWED
290 KFAIL = KFAIL + 1
IF (KFAIL - MFAIL) 300, 300, 260
C24.0 INCREMENT THE INTERMEDIATE FAILURE COUNTER, COMPUTE NEW DELTA T,
C AND RESTORE INITIAL CONDITIONS
300 KF = KF + 1
KHAI = 1
DT = DTM * QT
NN 310 T = 1, 17
310 S(11) = SS(11)
C25.0 SET THE INTERMEDIATE FAILURE COUNTER TO ZERO
320 KF = 0
GO TO 330
C19.0 COMPUTE NEW DT BASED ON THE VALUE OF THE ESTIMATED ERROR, ESTER
330 DT = FDT * DT * (ESTER ** 0.25
IF (DT / T - 1. E-0) 334, 334, 336
334 WRITE (6,335) DT

```

| EXTERNAL FORMULA NUMBER   | SOURCE STATEMENT | INTERNAL FORMULA NUMBER(S) |
|---|------------------|----------------------------|
| 1144 FORMATTED INPUT COMP INTERVAL = E17.8                            |                  | 09/01/65                   |
| GO TO 260   |                  | 7760H144                   |
| C20.0 TEST INFINITE FAILURE COUNTER                                   |                  | 7770H144                   |
| 136 IF (KFI) .331. 131. 340   |                  | 7775H144                   |
| C20.1 TEST PERTURBATION TOTAL FLAG                                    |                  | 7780H144                   |
| 331 GO TO 1332. 333). ICH   |                  | 7785H144                   |
| C20.2 PERTURBATION SOLUTION CALCULATIONS                              |                  | 7790H144                   |
| 332 KR = 5  |                  | 7795H144                   |
| C20.2.1 CHECK TIME AGAINST PRINT TIME                                 |                  | 7800H144                   |
| IF (T - TPI) 50. 331. 337   |                  | 7805H144                   |
| C20.2.2 INCREMENT PRINT TIME  |                  | 7810H144                   |
| 337 TPI = TPI + TPI   |                  | 7815H144                   |
| GO TO 220   |                  | 7820H144                   |
| C20.3 TEST IF THIS IS BOTH A PRINT AND A REGULAR COMPUTE POINT        |                  | 7830H144                   |
| C (IPRINT = 3)  |                  | 7835H144                   |
| 333 GO TO 1 50. 50. 332). IPRINT                                      |                  | 7836H144                   |
| C21.0 RESTORE VALUES SAVED IN STEP 7.1.5 TO THOSE IN STEP 7.1.6       |                  | 7840H144                   |
| 340 ON 350 I = 1. 17  |                  | 7845H144                   |
| 450 S (I11 = SS (I11  |                  | 7850H144                   |
| IPRINT = 2  |                  | 7855H144                   |
| GO TO 220   |                  | 7860H144                   |
| C26.0 SECOND PASS OF RUNGE-KUTTA                                      |                  | 7880H144                   |
| 360 KR = 3  |                  | 7890H144                   |
| DT4 = DT / 4.   |                  | 7900H144                   |
| HX = HX + DT4 * RK1X  |                  | 7910H144                   |
| HY = HY + DT4 * RK1Y  |                  | 7920H144                   |
| HZ = HZ + DT4 * RK1Z  |                  | 7930H144                   |
| RK2X = DT * HX20  |                  | 7940H144                   |
| RK2Y = DT * HY20  |                  | 7950H144                   |
| RK2Z = DT * HZ20  |                  | 7960H144                   |
| GO TO 230   |                  | 7970H144                   |
| C27.0 THIRD PASS OF RUNGE-KUTTA                                       |                  | 7980H144                   |
| 370 KR = 4  |                  | 7990H144                   |
| T = S(11) + DT  |                  | 8000H144                   |
| IS = S(11) + DT   |                  | 8010H144                   |
| TPSI = S(12) + DT   |                  | 8020H144                   |
| TNOD = S(13) + DT   |                  | 8030H144                   |
| TPER = S(14) + DT   |                  | 8040H144                   |
| GAMMA = S(17) + FP1 * DT  |                  | 8050H144                   |
| HX = S(12) + DT * S(5) + DT2 * RK2X                                   |                  | 8060H144                   |
| HY = S(13) + DT * S(6) + DT2 * RK2Y                                   |                  | 8070H144                   |
| HZ = S(14) + DT * S(7) + DT2 * RK2Z                                   |                  | 8080H144                   |
| RK3X = DT * HY2D  |                  | 8090H144                   |
| RK3Y = DT * HY2D  |                  | 8100H144                   |
| RK3Z = DT * HZ2D  |                  | 8110H144                   |
| GO TO 230   |                  | 8120H144                   |
| C28.0 FOURTH PASS OF RUNGE-KUTTA                                      |                  | 8130H144                   |
| 380 KR = 1  |                  | 8140H144                   |
| DT6 = DT / 6.   |                  | 8150H144                   |
| HX = S(12) + DT * S(5) + DT6 * (RK1X + RK2X + RK3X)                   |                  | 8160H144                   |
| HY = S(13) + DT * S(6) + DT6 * (RK1Y + RK2Y + RK3Y)                   |                  | 8170H144                   |
| HZ = S(14) + DT * S(7) + DT6 * (RK1Z + RK2Z + RK3Z)                   |                  | 8180H144                   |
| HND = S(5) + (RK1X + 2. * (RK2X + RK3X) + DT * HX2D)/6.               |                  | 8190H144                   |
| HYD = S(6) + (RK1Y + 2. * (RK2Y + RK3Y) + DT * HY2D)/6.               |                  | 8200H144                   |
| HZD = S(7) + (RK1Z + 2. * (RK2Z + RK3Z) + DT * HZ2D)/6.               |                  | 8210H144                   |
| C29.1 TEST WHETHER RESIDUAL PERTURBATION EQUATIONS OR TOTAL EQUATIONS |                  | 8220H144                   |
|   |                  | 8225H144                   |

|       | EXTERNAL FORMULA NUMBER  | SOURCE STATEMENT | INTERNAL FORMULA NUMBER(S) |
|-------|--|------------------|----------------------------|
| L     | OF MOTION AND BEING INTEGRATED                                 |                  | 8226H144 *207              |
|       | GO TO 1380. 381). ICH  |                  | 8230H144                   |
| C28.2 | TEST VALUE OF PAINT FLAG                                       |                  | 8235H144 *208              |
|       | 381 GO TO 1230. 382. 382A. IPRINT                              |                  | 8240H144                   |
| C28.3 | COMPUTE TWO INTERVAL INCREMENT IN VELOCITY                     |                  | 8245H144 *209              |
|       | 382 QFLX0 = QFLX0 + (RK1X +2.* (RK2X + RK3X) + DT * HX2D) / 6. |                  | 8250H144 *210              |
|       | QFLY0 = QFLY0 + (RK1Y +2.* (RK2Y + RK3Y) + DT * HY2D) / 6.     |                  | 8260H144 *211              |
|       | QFLZ0 = QFLZ0 + (RK1Z +2.* (RK2Z + RK3Z) + DT * HZ2D) / 6.     |                  | 8270H144                   |
| C29.0 | RETURN TO MAIN PROGRAM   |                  | 8275H144 *212              |
|       | 1230 RETURN  |                  | 8280H144 *213              |
|       | END  |                  | 8290H144 *214              |

Section 8  
SUBROUTINE PLOT

8.1 DEVELOPMENT OF EQUATIONS

The purpose of this subroutine is to plot the error in the Jacobi constant,  $\Delta C/C$ , for the analytic solution, for the numerical residual perturbation solution, for the precessing ellipse which forms the basis of the numerical perturbation solution, for a straightforward numerical solution of the total equations of motion, and for the initial osculating ellipse which would form the basis of an ordinary Encke method; and to plot the vector position and velocity differences of each of the other solutions from the numerical residual perturbation solution. The numerical residual perturbation solution is taken as the standard of comparison because it normally holds the Jacobi integral most constant. The routines are taken from reference 8.

The abscissa of the plots is time. The plotting symbols are o rasters wide and the length of the scales is 1,024 rasters less space required for labeling and scales. This allows about 152 points to a plot. Hand plots indicate a point about every quarter radian of true anomaly is necessary to describe the motion adequately. In the dimensionless variables, the mean motion is  $a^{-3/2}$  (ref 1, page 208). Thus, the time to be plotted per page to produce 152 points (corresponding to  $152/4 = 38$  radians) is given by:

$$t_R = (38a^{3/2})$$

To make the plot readable,  $t_R$  is rounded to have only one significant figure. If

$1 \times 10^n \leq t_R < 2 \times 10^n$ , then grid is drawn at intervals of  $0.1 \times 10^n$ ; if  $2 \times 10^n \leq t_R < 5 \times 10^n$ , then the grid is drawn at intervals of  $0.2 \times 10^n$ ; and if  $5 \times 10^n \leq t_R < 10^{n+1}$ , then the grid is drawn at intervals of  $0.5 \times 10^n$ .

## 8.2 EQUATIONS IN ORDER OF SOLUTIONS

### 8.2.1 Input

The input to subroutine PLOT is:

$a_{32}$  = The semi-major axis to the three-halves power  
 $T_f$  = Final time  
 $a$  = The semi-major axis  
 $e$  = The eccentricity  
 $i^{\circ}$  = The inclination in degrees  
 $\mu$  = The ratio of the mass of the smaller body to the sum of the masses of the two bodies  
 $C_{init}$  = The Jacobi constant  
 $P_{ss}$  = The anomalistic period of the precessing ellipse  
PTIME = Up to 700 values of time at which the data are to be plotted  
AP = A 9,100-element array defined as follows, (where  $i = 0, 1, 2, \dots, 699$  is the plot point count index):  
 $ap_{1+13i} = (\Delta C/C)_{ai}$  The subscripts are defined by:  
 $ap_{2+13i} = (\Delta C/C)_{pki}$   $a$  = analytic solution from reference 1  
 $ap_{3+13i} = (\Delta C/C)_{ci}$   $pk$  = precessing ellipse value  
 $ap_{4+13i} = (\Delta C/C)_{ki}$   $c$  = value obtained from numerical solution of total equations of motion

$$ap_{5+13i} = (\Delta C/C)_{pi} \quad k = \text{initial osculating ellipse value}$$

$$ap_{6+13i} = (\Delta p/a)_{ai} \quad p = \text{value from numerical residual}$$

$$\qquad \qquad \qquad \text{perturbation solution}$$

$$ap_{7+13i} = (\Delta p/a)_{pki}$$

$$ap_{8+13i} = (\Delta p/a)_{ci}$$

$$ap_{9+13i} = (\Delta p/a)_{ki}$$

$$ap_{10+13i} = (\Delta p \cdot a^{1/2})_{ai}$$

$$ap_{11+13i} = (\Delta p \cdot a^{1/2})_{pki}$$

$$ap_{12+13i} = (\Delta p \cdot a^{1/2})_{ci}$$

$$ap_{13+13i} = (\Delta p \cdot a^{1/2})_{ki}$$

### 8.2.2 Initial Setup

1. The machine is instructed to provide both 9 in. by 9 in. transparencies and 35mm slides.
2. The plotting symbols are chosen as:

$\circ$  = Analytic solution

$\times$  = Precessing ellipse

$\square$  = Numerical solution of total equations of motion

$*$  = Initial osculating ellipse

$\bullet$  = Numerical residual perturbation solution

3. Compute abscissa scale.

Set initial time for initial plot = 0.

A. Compute unrounded  $t_R$ :

$$t_R = 38. + a_{32}$$

B. Set number of characters to be displayed in horizontal labels:

$$NX = 3$$

C. Develop  $A = 10^n$  such that  $10^n \leq t_R \leq 10^{n+1}$

(1) Set  $A = 1$

(2) Compare  $t_R$  and 1:

If  $t_R < 1$ , go to Step 8.2.2-3 C(3).

If  $t_R = 1$ , go to Step 8.2.2-3 C(6).

If  $t_R > 1$ , go to Step 8.2.2-3 C(5).

(3) Develop  $A < 1$ :

Set  $A = A/10$  and increment digit count NX:

$$NX = NX+1$$

Compare  $t_R$  and A:

If  $t_R < A$ , repeat Step 8.2.2-3 C(3).

If  $t_R \geq A$ , go to Step 8.2.2-3 C(6).

(4) Set A = B and increment NX:

A = B

NX = NX+1

(5) Develop A > 1:

B = 10 A

Compare  $t_R$  and B:

If  $t_R \leq B$ , go to Step 8.2.2-3 C(6).

If  $t_R > B$ , go to Step 8.2.2-3 C(4).

(6) Compute integer portion of  $t_R/A = t_i$ :

Compare  $t_i$  and 2:

If  $t_i < 2$  (i.e., = 1), go to Step 8.2.2-3D.

If  $t_i = 2$ , go to Step 8.2.2-3E.

If  $t_i > 2$ , go to Step 8.2.2-3F.

D. Compute  $t_R$  and vertical line spacing for  $t_i = 1$ :

$\delta X = A/10$

$$t_R = 10\delta X t_i$$

Go to Step 8.2.2-3H.

E. Compute  $t_R$  and vertical line spacing for  $t_i = 2, 3$ , or  $4$ :

$$\delta X = A/5$$

$$t_R = 5\delta X t$$

Go to Step 8.2.2-3H.

F. Compare  $t_i$  and  $4$ :

If  $t_i \leq 4$  (i.e.,  $= 3$  or  $4$ ), go to Step 8.2.2-3E.

If  $t_i > 4$ , go to Step 8.2.2-3G.

G. Compute  $t_R$  and vertical line spacing for  $4 < t_i < 10$ :

$$\delta X = A/2$$

$$t_R = 2\delta X t_i$$

H. Print scale values:

$$t_R, A, \delta X, t_i, NX$$

4. The machine is instructed to use semi-log scales, log on vertical, and linear on horizontal.
5. The point count index II for beginning of plot point number is set  
 $II = 1$ . The page count IIK is set = 1.

#### 8.2.3 Computation for Each Set of Three Plots

1. Set point count index III for maximum end of plot point number  
 $III = II + 152$ .

2. Compute initial time EM and final time EN for each set of plots:

$$EM = EN$$

$$EN = EM + t_R$$

8.2.4 Set L = 1 to Indicate Plot for AC/C

8.2.5 Draw Grid and Do Common Labeling

1. Draw grid (subroutine GRIDV), calling sequence: general (L, XL, XR, YB, YT, DX, DY, N, M, I, J, NX, NY)

L = 1 indicates film to be advanced, job number, and frame count to be displayed

XL = EM is first time

XR = EN is last time

YB =  $10^{-10}$  is bottom of ordinate

YT = 1 is top of ordinate

DX =  $\delta X$  indicates vertical lines drawn at increments of  $\delta X$

DY = 1 indicates horizontal lines drawn at increments of 1

N = 0 causes all vertical lines to be of equal intensity

M = 2 with logarithmic ordinate, this is a dummy

I = -1 causes each vertical line to be labeled and labels to be below the scale

J = -4 with logarithmic ordinate, this is a dummy

NX = NX indicates NX significant places on X scale

NY = 1 indicates 1 significant place in Y scale

2. List and define symbols.

| <u>Symbol</u> | <u>Location</u><br>(Raster Counts) | <u>Definition</u>  | <u>Location</u><br>(Raster Counts) |
|---------------|------------------------------------|--|------------------------------------|
| o             | 125, 939                           | ANALYTIC SOLUTION  | 145-289, 939                       |
| x             | 125, 919                           | PRECESSING ELLIPSE<br>(REFERENCE ORBIT FOR<br>NUMERICAL RESIDUAL<br>PERTURBATION SOLUTION) | 145-408, 919<br>145-432, 899       |
| □             | 125, 879                           | NUMERICAL SOLUTION OF<br>TOTAL EQUATIONS   | 145-416, 879                       |
| *             | 125, 859                           | INITIAL OSCULATING<br>ELLIPSE  | 145-336, 859                       |

3. Label horizontal scale.

The word "TIME" goes in raster counts 475-507, 0.

4. List run parameters and time scale.

"SEMI-MAJOR AXIS = 'a' ECCENTRICITY = 'e' INCLINATION (DEGREES)  
= 'i' MASS RATIO = ' $\mu$ '" is printed in raster counts 162-942, 989.

"TIME RANGE FROM 'EN' TO 'EM'" is printed in raster counts 434-658,  
969.

5. If L = 1, go to Step 8.2.6.

If L = 2, go to Step 8.2.7.

If L = 3, go to Step 8.2.8.

### 8.2.6 Label and Plot Error in Jacobi Integral, ΔC/C

1. Label vertical scale and title.

"ABSOLUTE VALUE OF ΔC/C" is entered in raster counts 0, 800-348.

Label plot "ABSOLUTE VALUE OF NORMALIZED ERROR IN JACOBI INTEGRAL" in raster counts 302-727, 1012.

2. Add additional symbol definition and parameter values. The symbol '..' is printed at raster locations 125, 839.

"NUMERICAL RESIDUAL PERTURBATION SOLUTION" is printed in raster positions 145-465, 839.

"JACOBI CONSTANT, C = 'C<sub>init</sub>'" is printed at raster positions 132-348, 969.

"ANOMALISTIC PERIOD = 'P<sub>ss</sub>'" is printed at raster positions 718-950, 969.

3. Plot ΔC/C.

A. Set point count index i = II.

B. If  $t_i \leq t_f$ , go to Step 8.2.6-3C; otherwise go to Step 8.2.7.

C. Set index K to find location of  $(\Delta C/C)_{ai}$  in array AP:

$$K = -12 + 13i$$

D. If  $t_i \leq EN$ , go to Step 8.2.6-3E; otherwise go to Step 8.2.7.

E. Plot  $(\Delta C/C)_{ai}$ ,  $(\Delta C/C)_{pki}$ ,  $(\Delta C/C)_{ci}$ ,  $(\Delta C/C)_{ki}$ , and  $(\Delta C/C)_{pi}$ .

F. If  $i < III$ , set  $i = i+1$  and return to Step 8.2.6-3B; otherwise go to Step 8.2.7.

8.2.7 Label and Plot Position Differences from Numerical Residual

Perturbation Value,  $\Delta\rho/a$

1. Draw grid and do common labeling.

Set  $L = 2$  and go to Step 8.2.5.

2. Label vertical scale and title.

"DELTA RHO/A" is printed in raster locations 0, 602-446.

"NORMALIZED DISTANCE FROM NUMERICAL RESIDUAL PERTURBATION POSITION" is printed in raster locations 302-822, 1012.

3. Plot  $\Delta\rho/a$

A. Set point count index  $i = II$ .

B. If  $t_i \leq t_f$ , go to Step 8.2.7-3C; otherwise go to Step 8.2.8.

C. Set index  $K$  to find location of  $(\Delta\rho/a)_{ai}$ :

$$K = -7 + 13i$$

D. If  $t_i \leq EN$ , go to Step 8.2.7-3E; otherwise go to Step 8.2.8.

E. Plot  $(\Delta\rho/a)_{ai}$ ,  $(\Delta\rho/a)_{pki}$ ,  $(\Delta\rho/a)_{ci}$ , and  $(\Delta\rho/a)_{ki}$ .

F. If  $i < III$ , set  $i = i+1$  and return to Step 8.2.7-3B; otherwise go to Step 8.2.8.

**8.2.8 Label and Plot Velocity Differences from Numerical Residual Perturbation Value,  $\dot{\Delta\rho} \sqrt{a}$**

1. Draw grid and do common labeling. Set  $L = 3$  and go to Step 8.2.5.

2. Label vertical scale and title.

" $\dot{\Delta\rho} \sqrt{a}$ " is printed in rasters 0, 620-324.

"NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION" is printed in raster count 258-866, 1012.

3. Plot  $(\dot{\Delta\rho} a^{1/2})_{ai}$ ,  $(\dot{\Delta\rho} a^{1/2})_{pki}$ ,  $(\dot{\Delta\rho} a^{1/2})_{ci}$ , and  $(\dot{\Delta\rho} a^{1/2})_{ki}$ .

A. Set point count index  $i = II$ .

B. If  $t_i \leq t_{II}$  go to Step 8.2.8-3C; otherwise go to Step 8.2.10.

C. Set index K to find location of  $(\dot{\Delta\rho} a^{1/2})_{ai}$  in array AP:

$$K = -3 + 13i$$

D. If  $t_i < EN$ , go to Step 8.2.8-3E.

If  $t_i = EN$ , go to Step 8.2.8-3F.

If  $t_i > EN$ , go to Step 8.2.8-3G.

E. Plot  $(\dot{\Delta\rho} a^{1/2})_{ai}$ ,  $(\dot{\Delta\rho} a^{1/2})_{pki}$ ,  $(\dot{\Delta\rho} a^{1/2})_{ci}$ , and  $(\dot{\Delta\rho} a^{1/2})_{ki}$ .

If  $i < III$ , set  $i = i+1$  and return to Step 8.2.8-3B; otherwise set  $II = III$  and go to Step 8.2.9.

F. Plot  $(\dot{\Delta\rho} a^{1/2})_{ai}$ ,  $(\dot{\Delta\rho} a^{1/2})_{pki}$ ,  $(\dot{\Delta\rho} a^{1/2})_{ci}$ , and  $(\dot{\Delta\rho} a^{1/2})_{ki}$ .

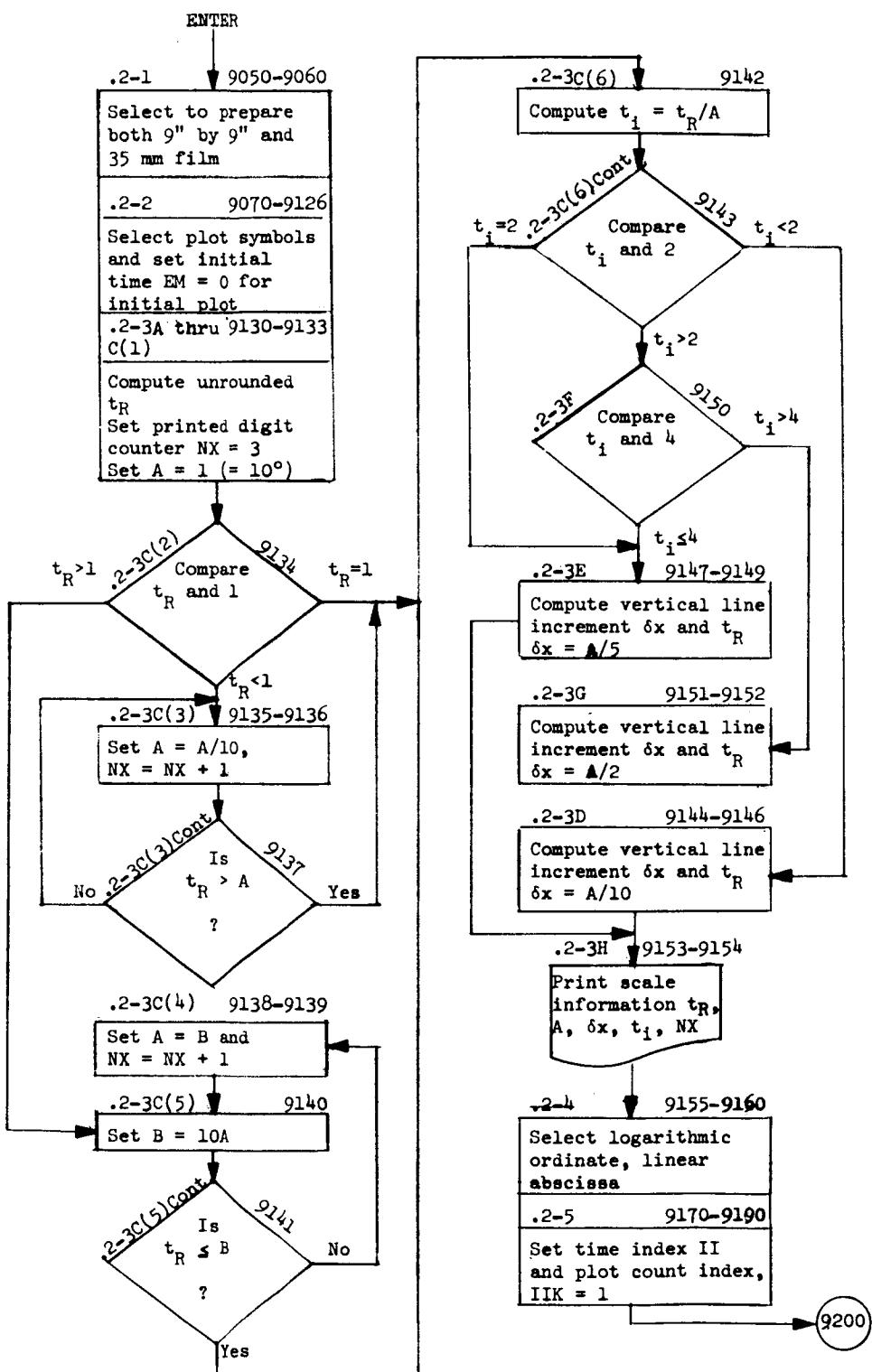
G. Set  $II = i$ .

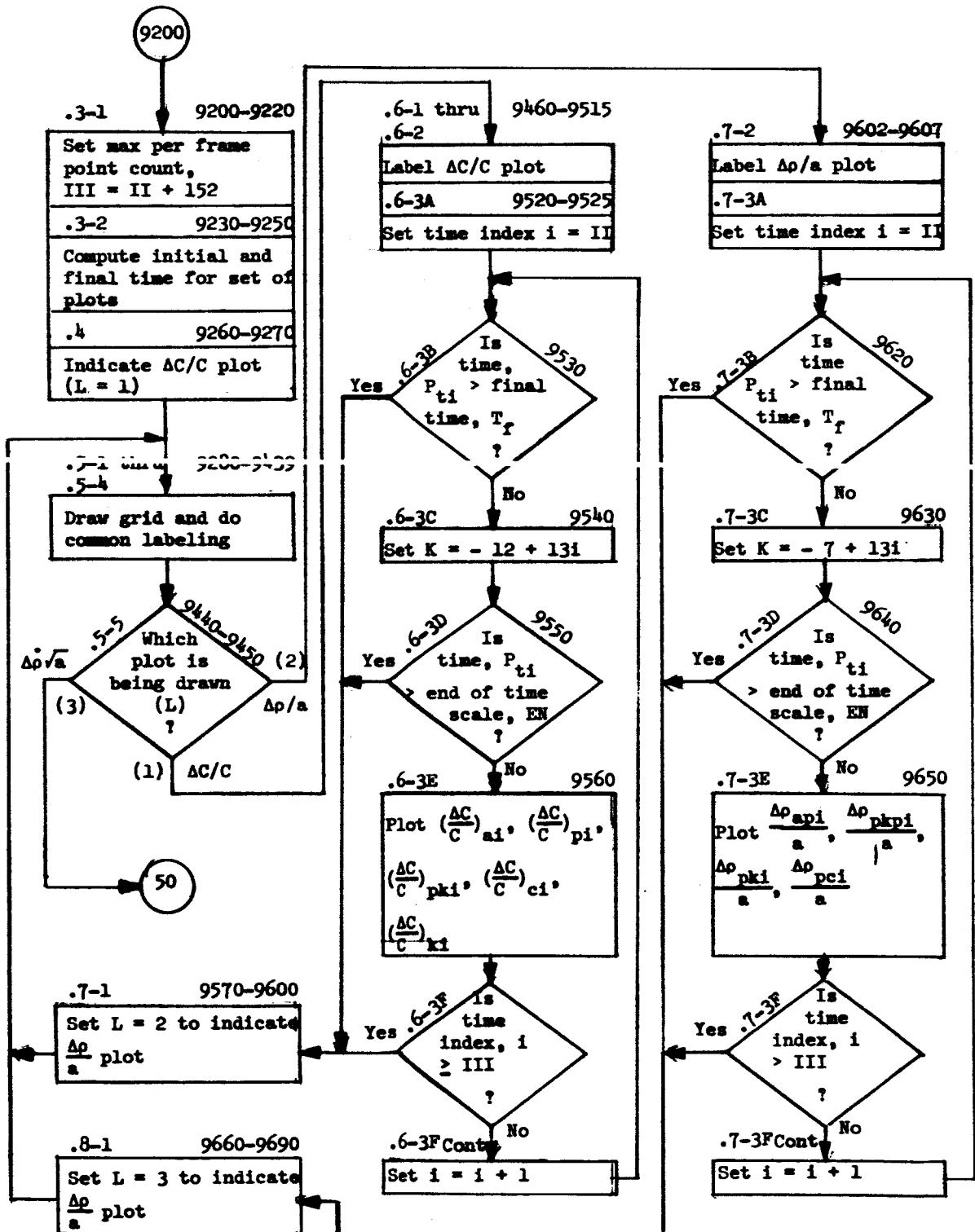
8.2.9 If  $IIK < 6$ , set  $IIK = IIK + 1$  and Return to Step 8.2.3-1; Otherwise Go To Step 8.2.10.

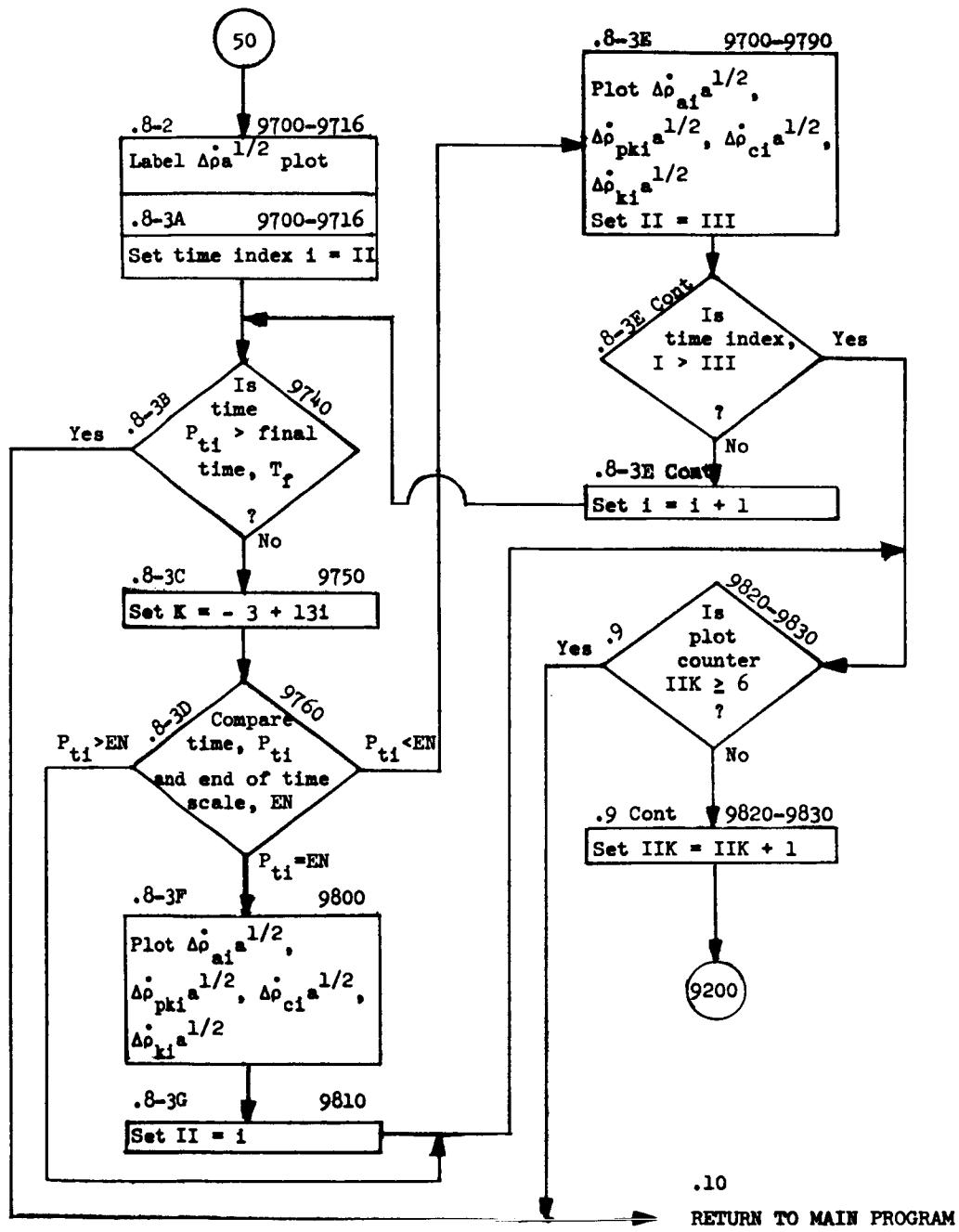
8.2.10 Return to Main Program

### 8.3 SUBROUTINE PLOT DETAIL FLOW CHARTS

The flow charts of this section describe the logic of subroutine PLOT completely, i.e., all conditional transfers are shown. The numbers to the top left of each box are the subsection numbers of Section 8.2 where the operations mentioned in the box are detailed. The numbers at the top right are the card numbers of the subroutine listing, Section 8.4.







#### 8.4 PLOT LISTING

This section contains the listing for subroutine PLOT. The subsection numbers on the comment cards refer to the subsections in Section 8.2, wherein the code is explained.

H144 PLOT - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S) PAGE 81  
 09/01/65

```

SUBROUTINE PLOT (A32,TF,A1,EC1,DEG1,EP1,JACOB1,PSS)
DIMENSION W(130),PTIME(700),AP(9100),MRPT(5)
COMMON W,IPRINT,TS,TPSI,TNOD,TPER,GAHNA,EPI,ITP,PTIME
1.AP
CR.2.2 INITIAL SETUP
CR.2.2-1 SFT FOR ROTH 9 IN AND 35 MM OUTPUT
CALL CARAV (935)
CR.2.2-2 SELECTION OF PLOTTING SYMBOLS
MRKPT (1) = 38
MRKPT (2) = 55
MRKPT (3) = 63
MRKPT (4) = 44
MRKPT (5) = 42
CR.2.2-3 COMPUTE ABSISSA SCALE
FN = 0.
TR = 38. * A32
NX = 3
A = 1.
210 IF (TR - A)220, 230, 240
220 A = A /10.
NX = NX + 1
IF11R -A) 220,230,230
250 A =R
NX = NX + 1
240 R = A * 10.
IF11R - A) 230,230,250
230 TI = AINI (TR / A)
IF11I - 2,) 260,270,280
260 DX = A /10.
TR = DX * 1C.. * TI
GO TO 300
270 CX = A/5.
TR = DX * 5.. * TI
GO TO 300
280 IF11I - 4,) 270,270,290
290 DX = A/2.
TR = DX * 2.. * TI
300 WRITE (6,310) TR, A, DX, TI, NX
310 FORMAT (1H 4E18.8, 18)
CR.2.2-4 SFT FOR LOGARITHMIC ORDINATE. LINEAR ABSISSA
CALL SHXYV (0, 1)
CR.2.2-5 SFT INITIAL FLAGS
L=1
00 15G L1K=1.6
CR.2.3 COMPUTATION FOR EACH SET OF 3 PLOTS
CR.2.3-1 SFT MAXIMUM PER FRAME POINT COUNT
L1 = L1 + 152
CR.2.3-2 COMPUTE INITIAL AND FINAL TIME FOR EACH SET OF PLOTS
EN = FN
EN = EN + TR
CR.2.4 SET L = 1 TO INDICATE PLOT FOR DELTA C/C
L=1
CR.2.5 DRAW GRID AND DO COMMON LABELING
CR.2.5-1 DRAW GRID (SUBROUTINE GRIDV)
20 CALL GRIDV (1.E4,EN,1.E-10,1.0,DX,1.0,0.2,-1,-4,NX,-1)

```

09/01/65  
 INTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER  
**H144 PLOT**  
**EXTERNAL FORMULA NUMBER** - **SOURCE STATEMENT** - **INTERNAL FORMULA NUMBER**  
**C8.0-2-9-2 LIST AND OFFINE SYMBOLS**  
 CALL PRINIV(125,919,381)  
 CALL PRINIV(116,18H ANALYTIC SOLUTION,145,939)  
 CALL PRINIV(125,919,351)  
 CALL PRINIV(139,94H PRECESSING ELLIPSE (REFERENCE ORBIT FOR,137,91919350H,144  
 CALL PRINIV(141,61H NUMERICAL RESIDUAL PERTURBATION SOLUTION,145,897091930H,144  
 191  
 CALL PRINIV(125,679,631)  
 CALL PRINIV(138,38H NUMERICAL SOLUTION OF TOTAL EQUATIONS,137,8791930H,144  
 CALL PRINIV(125,659,441)  
 CALL PRINIV(122,27H INITIAL OSCULATING ELLIPSE,137,859)  
 CALL PRINIV(3-1 LABEL HORIZONTAL SCALE  
 CALL PRINIV(4,4H,AH, 475, 0 )  
**C8.0-2-5-4 LIST RUN PARAMETERS AND TIME SCALE**  
 CALL PRINIV(17,11HSENI-MAJOR AXIS =, 132, 969 )  
 CALL LABLY(1A1,268,989,7,1,1)  
 CALL PRINIV(14,1HNECCENTRICITY =, 340, 989)  
 CALL LABLY(1C1,452,989,7,1,0)  
 CALL PRINIV(2,2H,2HINCLINATION (DEGREES) =, 524, 989)  
 CALL LABLY(1D1,710,989,7,1,1)  
 CALL PRINIV(1,12,12HMAS RATIO =,  
 CALL LABLY(EP4,980,980,-3,1,1)  
 CALL PRINIV(15,15H TIME RANGE FROM,434,961)  
 CALL LABLY(1F1,362,969,4,-1,1)  
 CALL PRINIV(2,2H0,662,969,1)  
 CALL LABLY(EM1,626,969,4,-1,31)  
**C8.0-2-5-5 DETERMINING WHICH PLOT IS BEING DRAWN**  
 GO TO 130,40,501  
**C8.0-2-6 LABL AND PLCT ERROR IN JACOBI INTEGRAL.**  
**C8.0-2-6-1 LABEL VERTICAL SCALE AND TITLE**  
**C8.0-2-6-2 ADDITIONAL PLOT DEFINITION AND PARAMETER VALUES**  
 CALL PLOT(125,839,421)  
 CALL PRINIV(19,19HJACOBI CONSTANT,C = \* 132,969)  
 CALL LABLY(JACOB1,292,969,7,-1,6)  
 CALL PRINIV(110,71H ANHOLISTIC PERIOD =718,969)  
 CALL LABLY(IPSS,89,981,7,1,2)  
 CALL PRINIV(40,0HNUMERICAL RESIDUAL PERTURBATION SOLUTION,145,899510H,144  
 11  
**C8.0-2-6-3 PLOT DELTA C/C**  
 DIFC1 = 11.111  
 DIFP1 = 11.111  
 10 K = 11+1E-1  
 IF(P1M111 = ENA 90,90,100)  
 GO CALL APLOT(15,OPTIMIZ, APIK1, 01,5,MNPRT,IER1)  
**C8.0-2-7 LABL AND PLOT POSITION DIFFERENCES FROM NUMERICAL RESIDUAL**  
 PEAKING VALUE DELTA RHO/A  
**C8.0-2-7-1 CHAN GRID AND QD COMMON LABELING**  
 100 LG 1C,20  
**C8.0-2-7-2 LABEL VERTICAL SCALE AND TITLE**  
 40 CALL APLOT(10-1,2, 13,13HDLTA RHO / A=0.602)  
 CALL PRINIV(165,65HNORMALIZED DISTANCE FROM NUMERICAL RESIDUAL PER606H,144  
 ITURATION POSITION, 3021021

PAGE 83  
 09/01/65  
 INTERNAL FORMULA NUMBER(S)  
 PLOT EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)

```

M144 PLOT DELTA RHOA
C8.2-7-3 PLOT DELTA RHOA
      DO 60 I = 1,111
      IF(IPTIME(1)-TF1130,130 .GT. 110
      130 K = -7 + 13*I
      IF(IPTIME(1) - EN160,00,110
      140 CALL APIOTY (4,PITIME(1),AP(K),0,1,4,MRKPT,ERR)
      C8.2-8 LABL AND PLOT VELOCITY DIFFERENCES FROM NUMERICAL RESIDUAL
      C8.2-8 PERTURBATION VALUE. DELTA RHO DOT SQ RT A
      C8.2-8-1 DRAW GRID AND DO COMMON LABELING
      110 L=3
      GO TC 20
C8.2-8-2 LABEL VERTICAL SCALE AND TITLE
      50 CALL APRTIV (0,-12,2,27)DELTA RHO DOT TIMES SQ RT A. 0. 6201
      CALL PRNTV (76,76)NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL R9715H144
      50 CALL APRTIV (0,-12,2,27)DELTA RHO DOT * SQ RT A. 0. 6201
      CALL PRNTV (76,76)NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL R9715H144
      C8.2-8-3 PLOT (DELTARHO DOT * SQ RT A) SUB A1. (DELTARHO DOT * SQ
      R1 A1 SUB PK1. (DELTARHO DOT * SQ RT A1) SUB A1. (DELTARHO DOT * SQ
      RCT * SQ RT A1) SUB C1. AND (DELTARHO DOT * SQ RT A1) SUB C1. AND (DELTARHO DOT * SQ
      C DO TC 1 = 1,111
      C
      IF (IPTIME(1) - TF1120,120,80
      120 K = -3 + 13 * I
      IF (IPTIME(1)-EN)70,160,140
      70 CALL APIOTY (4,PITIME(1),AP(K),0,1,4,MRKPT,ERR)
      II = III
      GO TC 150
      160 CALL APIOTY (4,PITIME(1),AP(K),0,1,4,MRKPT,ERR)
      140 II = 1
      C1.2-9 IF IIK IS LESS THAN 6, SET IIK = IIK + 1 AND RETURN TO STEP
      C B.2-3-1 OTHERWISE GO TO STEP B.2-10
      150 CONTINUE
      C8.2-10 RETURN TO MAIN PROGRAM
      80 RETURN
      END
  
```

Section 9  
INPUT AND OUTPUT

### 9.1 INPUT

The input to this program is by the Douglas subroutine INPUT 1. A load sheet for this program is shown as figure 2. The card format is:

column 1 - a one  
columns 2 through 6 - location number of piece of data  
columns 7 through 15 - input number  
columns 16 and 17 - location of decimal place from beginning of field  
positive if to the right

Three other pieces of data may be entered on the card. The location numbers are punched in columns 18-22, 34-38, and 50-54. The data are punched in columns 23-31, 39-47, and 55-63. The exponents, as explained above, are punched in columns 32-33, 48-49, and 64-65, respectively. The remaining information required is:

columns 66-68, zeros  
columns 69-70, reference run number  
columns 71-73, case number.

This routine allows identification on the card of each piece of input data by relative location number; only-non-zero numbers need be entered. It has a "Reference Run," "Case" setup. If the case number (card columns 71 to 73) is non-zero, but the reference run number (card columns 69, 70) is zero, then the data on the load sheet are assumed to be sufficient and the case is computed. If the case number is zero and the reference run number is non-zero, the data are stored in array WR and no case is attempted. If the following load sheets with non-zero case numbers have also the reference run number of the stored array, then a case is run using the input of array WR as modified by the new load sheet.

**FORTRAN DATA LOAD SHEET**

Form 60-720-1 (11-62)

PREPARED BY \_\_\_\_\_ DATE \_\_\_\_\_ PAGE \_\_\_\_ OF \_\_\_\_

#### ENGINEER'S NOTE:

MUST BE FILLED IN FOR  
PROPER PROCESSING

**PROBLEM** \_\_\_\_\_

PAGE \_\_\_\_\_ OF \_\_\_\_\_

LOCATION 00000 MUST NOT BE USED

59-7071-2

**KEY PUNCH STANDARD DATA INPUT**

Figure 2

The order of stacking cases is then:

1. All cases with zero reference run number
  2. First reference run (zero case number)
  3. All cases with first reference run number and non-zero case number
  4. Second reference run (zero case number)
- •  
•

It is one of the peculiarities of this routine that cases must be run in numerical order.

## 9.2 OUTPUT

The first information that is printed each time the program is run consists of an explanation of the error codes. The program makes certain checks on the input to make sure that the input is reasonable and consistent with the theory being used. Throughout the program there are additional checks made to ascertain if division by zero has occurred which would cause the program to produce nonsense if the case continued. In these cases, an error code is printed and the case terminated. The print is as follows:

| ERROR CODE | REASON FOR HALT   |
|------------|---|
| 1          | SEMI-MAJOR AXIS = 0   |
| 2          | ECCENTRICITY EQUALS OR EXCEEDS 1                                  |
| 7          | S (THE APPROX RADIUS VECTOR) = 0                                  |
| 8          | DERIVATIVE OF TIME WRT PSI = 0<br><br>(i.e., $\frac{dt}{d\psi}$ ) |

The error code is set to 1 in Section 4.3.2.1.1. The error code is set to 2 in Section 4.3.2.2.1. The error code is set to 7 in Section 4.17.3. The error code is set to 8 in Section 4.17.8.1. The error code is printed in Section 4.3.2.1.2.

The program then reads in a case and INPUT 1 prints:

REFERENCE RUN NO. (RR#) CASE NO. (CASE #)

There follows a floating point print of the input array W and of constants computed by the program as described in Section 4.4.1.

Section 4.6 then prints the headings for the print during the numerical residual perturbation solution as follows:

| THE INITIAL CONDITIONS ARE |         | SEMI-MAJOR AXIS a | INCLINATION i    |
|----------------------------|---------|-------------------|------------------|
|                            |         | ECCENTRICITY e    | MASS RATIO $\mu$ |
| TIME                       | DELTA X | DELTA X DOT       | DELTA X DBL DOT  |
|                            | DELTA Y | DELTA Y DOT       | DELTA Y DBL DOT  |
| C APPROX                   | DELTA Z | DELTA Z DOT       | DELTA Z DBL DOT  |
| X                          | X DOT   | X APPROX          | X DOT APPROX     |
| Y                          | Y DOT   | Y APPROX          | Y DOT APPROX     |
| C EXACT                    | Z       | Z DOT             | Z DOT APPROX     |

The algebraic interpretation of these headings is given in Section 4.10.

The initial value of the Jacobi constant,  $C_{init}$ , is printed in Section 4.8.6 as:

THE JACOBI INTEGRAL CONSTANT IS ( $C_{init}$ )

During computation of the numerical residual perturbation solution, the printing according to the above heading is accomplished in Section 4.10. If the input quantity  $W_{13} > 0$ , then printing takes place about every 1/4 radian of  $\psi$ . Otherwise printing takes place at the first and last point only.

The headers for the identification of the print during the numerical solution of the total equations of motion are printed in Section 4.19.4 and are as follows:

## NUMERICAL SOLUTION OF TOTAL EQUATIONS OF MOTION DATA

| TIME             | X | X DOT | X DBL DOT | RHO     |
|------------------|---|-------|-----------|---------|
|                  | Y | Y DOT | Y DBL DOT | RHO DOT |
| JACOBI CONSTANT, | Z | Z DOT | Z DBL DOT |         |

The printing for the numerical solution of the total equations of motion is done in Section 4.20 and the explanation of the above headers is shown in that section. Additional print for which there are no headers is also shown there. Printing during the solution of the total equations of motion has the same rules with regard to the input quantity  $w_{13}$  as does the print during the numerical residual perturbation solution.

The plotted output is described in subroutine PLOT, Section 8.

## SAMPLE PRINTED OUTPUT

| ERROR CODE | REASON FOR HALT                  |
|------------|----------------------------------|
| 1          | SEMI-MAJOR AXIS = 0              |
| 2          | ECCENTRICITY EQUALS OR EXCEEDS 1 |
| 3          | THE APPROX RADIUS VECTOR = 0     |
| 4          | DERIVATIVE OF TIME WRT PST = 0   |

| REFERENCE RUN NO. | 3               | CASE NO.        | 90              |                 |                 |
|-------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.1478CC00E 01    | 0.              | 0.              | 0.              | 0.30019999E-05  | 0.99000000E-00  |
| 0.2CC0000E-06     | 0.40000000E 03  | 0.69999999E-01  | 0.15000000E 01  | 0.09999999E-02  | 0.              |
| 0.50000000E 03    | 0.99999999E-08  | 0.              | 0.              |                 |                 |
| -0.1693162E 01    | -0.16471118E 01 | -0.12474339E 01 | 0.92941266E 00  | 0.89042391E 00  | -0.23349352E-02 |
| 0.                | 0.              | 0.              | -0.00000000E-19 | -0.00000000E-19 | 0.09999999E 01  |
| 0.9992CC19E-05    | 0.99999700E 00  | 0.17968492E 01  | 0.70567333E-01  | 0.85882531E 00  |                 |
| 0.46791927E-02    | -0.23349398E-02 | -0.10917198E-04 | 0.27315607E-04  |                 |                 |

THE INITIAL CONDITIONS ARE                    SEMI-MAJOR AXIS            0.14780000E 01  
 ECCENTRICITY                                0.  
 INCLINATION,DEG                            0.  
 MASS RATIO                                  0.3001999E-05

| TIME                            | DELTA X         | DELTA X DOT     | DELTA X DBL DOT | RHO            |
|---------------------------------|-----------------|-----------------|-----------------|----------------|
|                                 | DELTA Y         | DELTA Y DOT     | DELTA Y DBL DOT | RHO DOT        |
| C APPROX                        | DELTA Z         | DELTA Z DOT     | DELTA Z DBL DOT | RHO            |
| X APPROX                        | X DOT           | X APPROX        | X DOT APPROX    |                |
| C EXACT                         | Y               | Y DOT           | Y DOT APPROX    |                |
| Z                               | Z DOT           | Z APPROX        | Z DOT APPROX    |                |
| THE JACOBI INTEGRAL CONSTANT IS | -0.11590964E 04 |                 |                 |                |
| 0.                              | 0.              | 0.              | 0.89611433E-02  | 0.14780000E 01 |
|                                 | 0.              | -0.26841874E-02 | 0.              | 0.82332655E 00 |
| 0.                              | 0.              | -0.             | 0.              |                |
|                                 | 0.14780000E 01  | 0.82332655E 00  | 0.14780000E 01  | 0.82332655E 00 |
| 0.                              | 0.              | 0.              | 0.              | 0.             |

INTEGRATION OF COHESIVE CONSENSUS OF MOTION

MAXIMUM DELTA E PERTURBED = 0.11920929E-04

Section 10  
DISCUSSION OF RESULTS

A summary of numerical results is included in graphical plot form in the Appendix. For the numerical studies, the mass ratios were  $\mu = 3 \times 10^{-6}$ , which corresponds to the sun-earth (or quite closely to the sun-Venus) systems and  $\mu = 0.0123$ , which corresponds to the earth-moon system. In reference 1 it was stated that the solutions should be valid for a region of order  $\mu^{1/2}$  around the origin in the natural variables, i.e., semi-major axes of order unity in the blown-up variable could be tolerated. This would represent the limit to which the series could be taken. Further approximation is involved in that the actual series are only computed to order  $\epsilon^2$ . All  $\epsilon^2 e$  terms were dropped in this program for consistency since the switch-back terms from  $\epsilon^3$  (presumed to be of order  $\epsilon^2 e$ ) and terms of forms  $\epsilon e i$  and  $\epsilon^3$  are not included in reference 1. For the earth-moon mass ratio, semi-major axes of 0.041 (corresponding to a near grazing lunar satellite), 0.1, and 0.4 were used. For the earth-sun mass ratio, the semi-major axes were chosen to be 0.15 (corresponding to about the synchronous altitude), 0.5, and 1.478 (corresponding to the moon). In reference 1 the values of eccentricity and inclination that could be tolerated consistently with the accuracy with respect to  $\mu$  were not specified explicitly, but it was merely stated that these elements must be small. We have used eccentricities of 0 and 0.05 and inclinations of  $0^\circ$  and  $5^\circ$ . The plots show the time history of the error in the Jacobi constant, the vector distance from the numerical residual perturbation value, and the vector velocity difference from the numerical residual perturbation value. The summary of the cases computed is given in Table I.

The error in the Jacobi constant is represented as the variation with time along the solution, divided by the initial value. It is presented for the analytic solution of reference 1, for a precessing mean Kepler ellipse which has essentially the same precession rate as the analytic solution but no oscillatory perturbations, for a numerical integration of the total equations of motion, for the fixed initial osculating ellipse, and for the numerical residual perturbation solution. The precessing ellipse is the reference orbit of the numerical residual perturbation solution. The fixed initial osculating

ellipse is provided for comparison since it would form the basis of a normal Encke's method solution. The variation of the Jacobi constant should, of course, be zero in the exact solution of the restricted three-body problem. The other two variables were chosen to represent position and velocity errors, and were referenced to the numerical residual perturbation solution since it was anticipated that this solution would, in general, maintain the Jacobi integral most accurately. For each set of parameters, the progress of these results was monitored for approximately 28 revolutions. This limit is set by the program which, by the way it is coded, imposes a storage limit at this level. Since it was desired to show both the short- and long-term performance of the technique, an expanded time scale was used. This causes a plot for 28 revolutions to extend over five to six pages. For the 24-hour circular earth orbit and the moon's orbit, the entire time span is exhibited. For the other cases, only the beginning and ending pages are included.

It will be seen from the error in Jacobi integral plots that for the 24-hour circular earth orbit (or 13,000-mile Venus orbit) and the grazing lunar orbit, the numerical residual perturbation solution approximates the sun's perturbations no better than the analytic solution, which is already at the round off level of the machine throughout the 28 orbits. It will also be noted that for these cases, the initial osculating ellipse maintains the Jacobi integral better than the particular numerical solution of the total equations of motion used. It will be noted that in most cases the initial osculating ellipse holds the Jacobi integral better than the precessing mean-ellipse. However, the position and velocity errors of the initial osculating ellipse are greater than (sometimes orders of magnitude greater) or equal to the precessing ellipse errors. This may be explained as follows. For motion of a close satellite around the smaller primary, the Jacobi integral is sensitive to errors in radius and its derivative. It is almost totally insensitive to errors along the direction of motion. The  $s$ , or one-over-radius, expression in the analytic solution contains a constant term in the perturbations of order  $\epsilon^2$  showing that the radius vector oscillates about a value which is not the Keplerian value of the precessing ellipse but is actually less. The  $dt/d\psi$  expression contains a perturbation term whose value is not zero at

initial time. This term is used in the computation of the elements of the osculating ellipse but not in the computation of the elements of the mean ellipse. The result is that the period of the osculating ellipse is in error, but the perigee is lowered, partially compensating for the neglected term in  $s$ . It is thus better in radius but poorer in angle.

At the largest semi-major axes considered (the case of the moon about the earth, and the 10 lunar radii lunar satellite), the analytic solution is very poor. This would be anticipated since de Pontecoulant demonstrated in the last century that this series was, at best, only barely convergent for the moon and only second order terms have been carried in this study. Over half of the precession of perigee and over half the "evection" were lost when the higher order terms were dropped (ref 3, pages 322 and 326). At these semi-major axes, the advantage of the numerical residual perturbation solution without rectification over the numerical solution of the total equations of motion in holding the Jacobi integral drops to a factor of 6 to 7. However, the sharp difference in slope between the two values of error in the Jacobi constant for small times suggests that with rectification, the numerical residual perturbation solution will be much better than the numerical solution of the total equations of motion. But since the  $\Delta p/a$  and  $\Delta \dot{p}/\sqrt{a}$  plots show that the initial osculating ellipse is nearly as good as the precessing mean ellipse at small times, the advantage of the numerical residual perturbation solution over a numerical perturbation solution is not so clear. This suggests that when frequent rectification is required, as it seems to be with the large orbits, the use of a precessing ellipse whose initial orbital parameters are equal to those of an initial osculating ellipse might be better than using a precessing mean ellipse.

Comparing the results for  $e = 0.05$  with the results for  $e = 0$ , it seems that  $e = 0.05$  is a large  $e$  for this expansion. However, this program was constructed merely for demonstration of the residual perturbation method and is not deemed to be a practical analysis tool. In reference 6, the theory is developed without the restriction of small eccentricity or small inclination. The accuracy shown here for the case of zero inclination

and zero eccentricity then would be the only one applicable to a program based on that theory, and it would be expected that it would hold to the same order of accuracy at other eccentricities and inclinations. From the cases with zero-eccentricity, it appears that the assumption that the precessing ellipse would be nearly as useful a reference orbit as the analytic solution is not a good one. In hind sight it is seen that this would follow from the fact that in an accumulation, as is done in the numerical solution of differential equations, the decimal place where error accumulates is proportional to the maximum value of the solution. Thus, when an oscillation is being accumulated numerically, the error accumulates at a rate roughly proportional to the full magnitude of the oscillation. If the method of reference 6 were programmed, then it is anticipated that the additional algebra required to algebraically cancel all  $\epsilon e$  terms and  $\epsilon^2$  in the residual perturbation equations of motion would result in appreciably improved numerical results.

Table I. - SUMMARY OF COMPUTED CASES

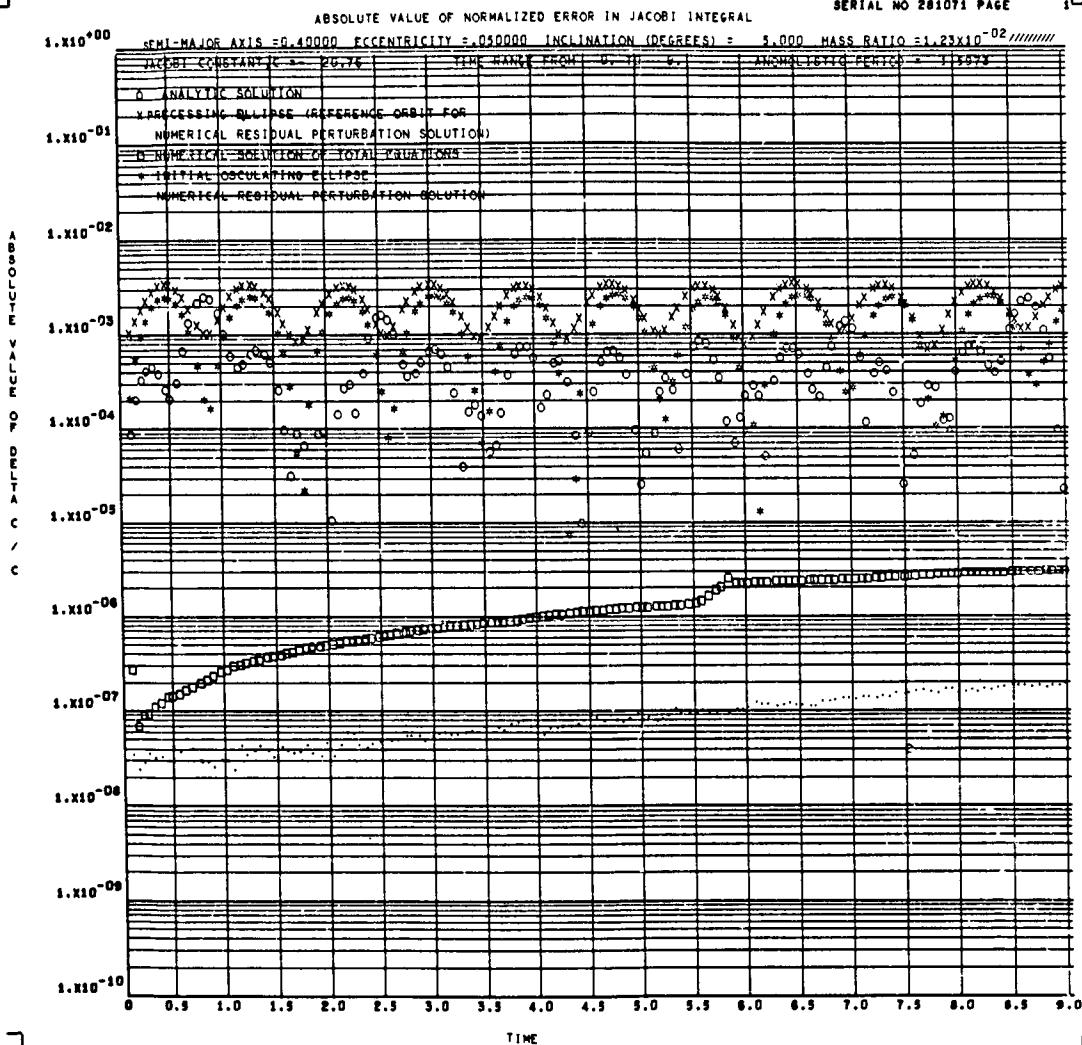
| $\mu$                 | Mass Ratio | Semi-major Axis<br>$a$ | Eccentricity<br>$e$ | Inclination<br>$i^\circ$          | Notes  | Frame Identification<br>Serial Number | Page Numbers        |
|-----------------------|------------|------------------------|---------------------|-----------------------------------|--|---------------------------------------|---------------------|
| $3 \times 10^{-6}$    | 0.15       | 0                      | 0                   | 0                                 | Synchronous<br>earth or 13,000-<br>mile Venus<br>satellite | 281071                                | 145-162             |
|                       | 0.15       | 0.05                   | 5                   |                                   |  | 281071                                | 127-129,<br>139-144 |
|                       | 0.5        | 0.05                   | 5                   |                                   |  | 281071                                | 109-111,<br>124-126 |
|                       | 1.478      | 0                      | 0                   |                                   |  | 302074                                | 1-3,<br>13-18       |
|                       | 1.478      | 0.0549                 | 5.15                |                                   | Moon's orbit or<br>120,000-mile<br>Venus satellite         | 281071                                | 91-108              |
| $1.23 \times 10^{-2}$ | 0.041      | 0                      | 0                   |                                   | Grazing lunar<br>satellite                                 | 281071                                | 55-57,<br>67-69     |
|                       | 0.041      | 0.05                   | 5                   |                                   |  | 281071                                | 37-39,<br>49-51     |
|                       | 0.1        | 0.05                   | 5                   |                                   |  | 281071                                | 19-21,<br>34-36     |
|                       | 0.4        | 0.05                   | 5                   | 10 lunar radii<br>lunar satellite |  | 281071                                | 1-3,<br>13-15       |

#### REFERENCES

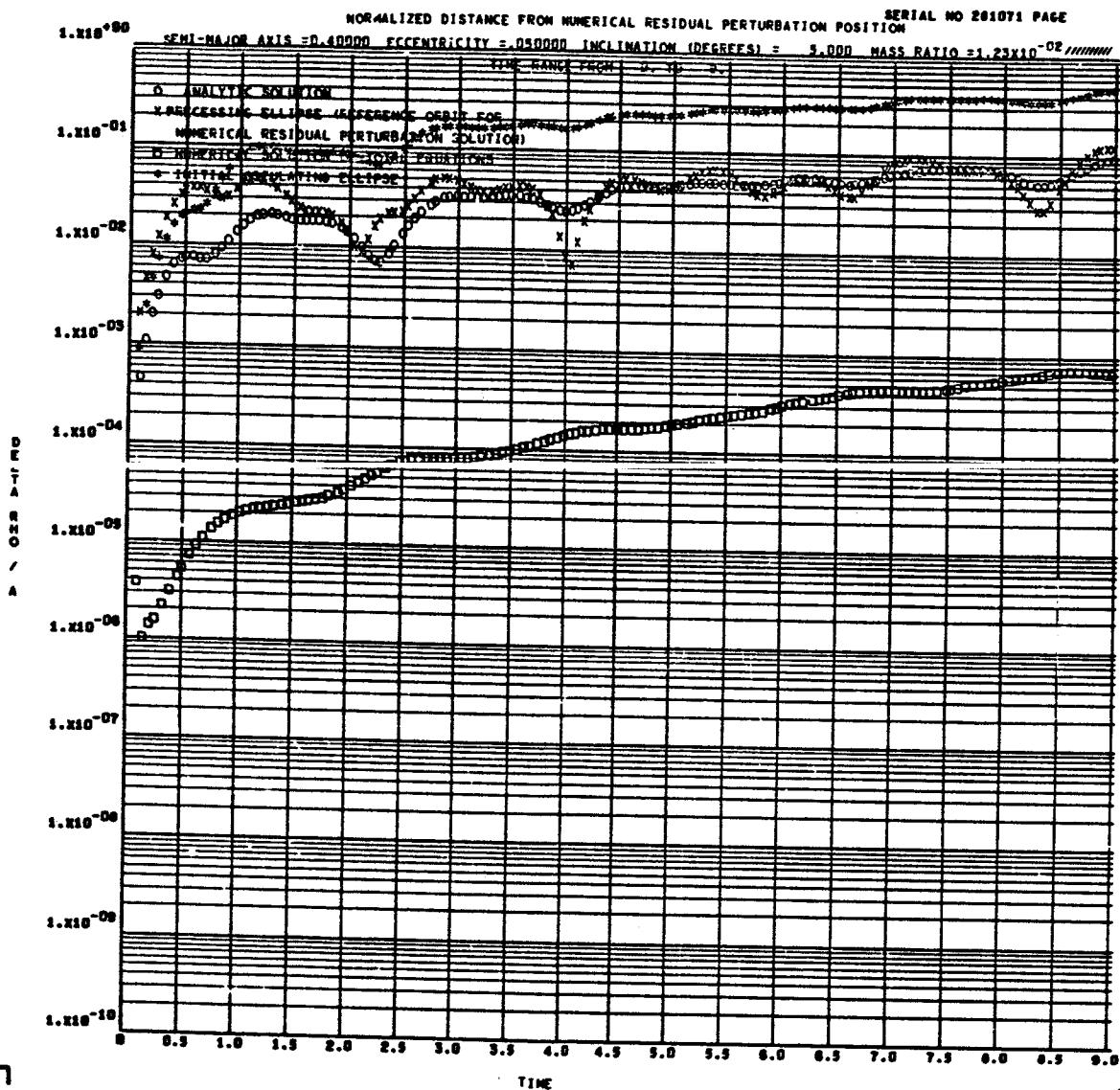
1. Kevorkian, J.: A Uniformly Valid Asymptotic Representation for All Times of a Satellite in the Vicinity of the Smaller Body in the Restricted Three-Body Problem. *Astron. J.*, vol. 67, no. 4, no. 1299, May 1962 (Revised version of reference 7.)
2. Moulton, Forest Ray: An Introduction to Celestial Mechanics. The Macmillian Company, 1914.
3. Brouwer, Dirk; and Clemence, Gerald M.: Methods of Celestial Mechanics. Academic Press, 1961.
4. Herget, Paul: The Computation of Orbits. 1948.
5. Hildebrand, F. B.: Introduction to Numerical Analysis. McGraw-Hill Book Co., Inc., 1956.
6. Eckstein, M.; Shi, Y; and Kevorkian, J.: Satellite Motion for Arbitrary Eccentricity and Inclination Around the Smaller Body in the Restricted Three-Body Problem. Douglas Paper No. 3489, April 1965.
7. Kevorkian, J.: A Uniformly Valid Asymptotic Representation for All Times of a Satellite in the Vicinity of the Smaller Body in the Restricted Three-Body Problem. Douglas Paper No. 1163, May 1961.
8. Anon.: North American Aviation, Inc., Engineers Computing Manual, Region 74, S-C 4020 Subprograms, Oct. 1963.

**APPENDIX**  
**SAMPLE GRAPHICAL RESULTS**

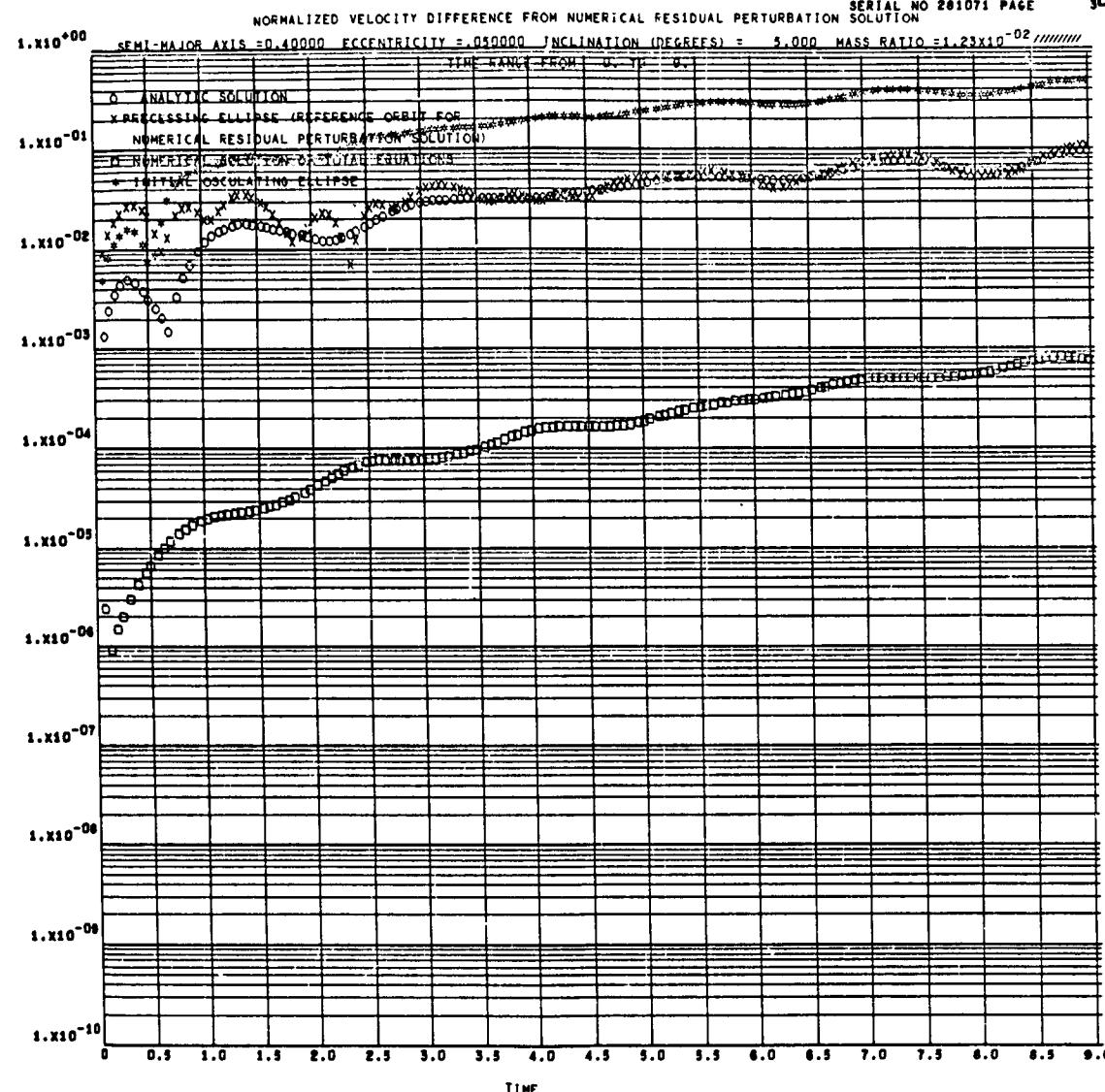
SERIAL NO 281071 PAGE



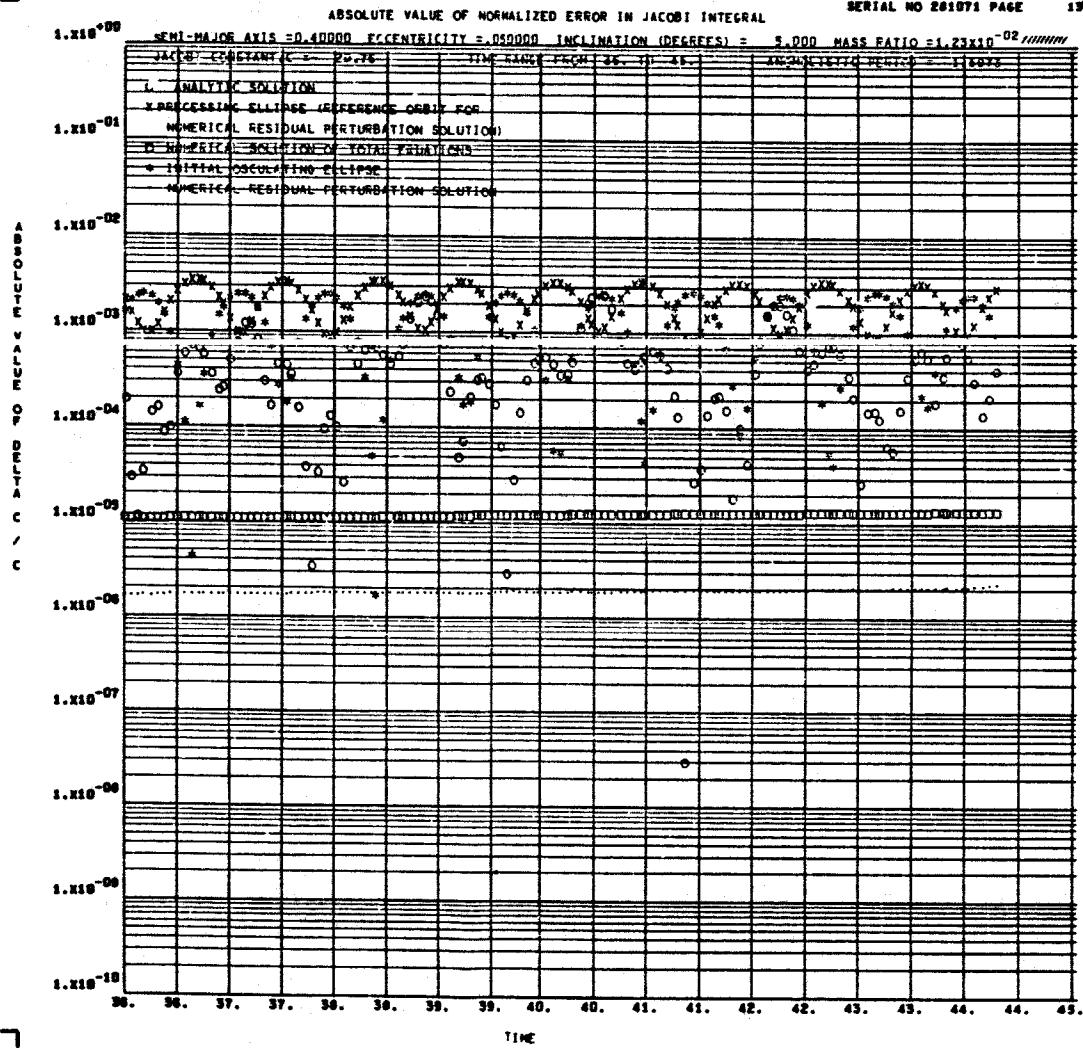
SERIAL NO 281071 PAGE



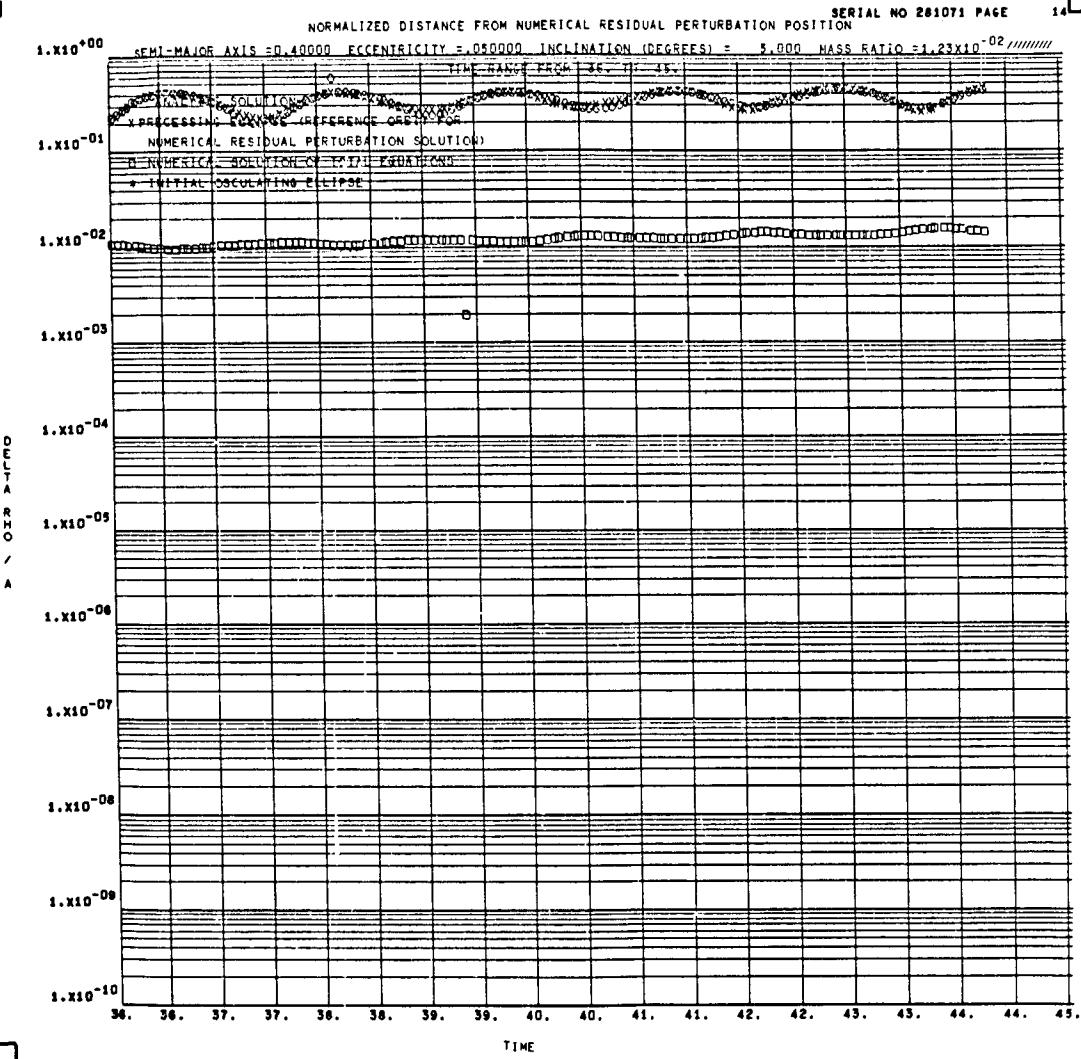
SERIAL NO 281071 PAGE



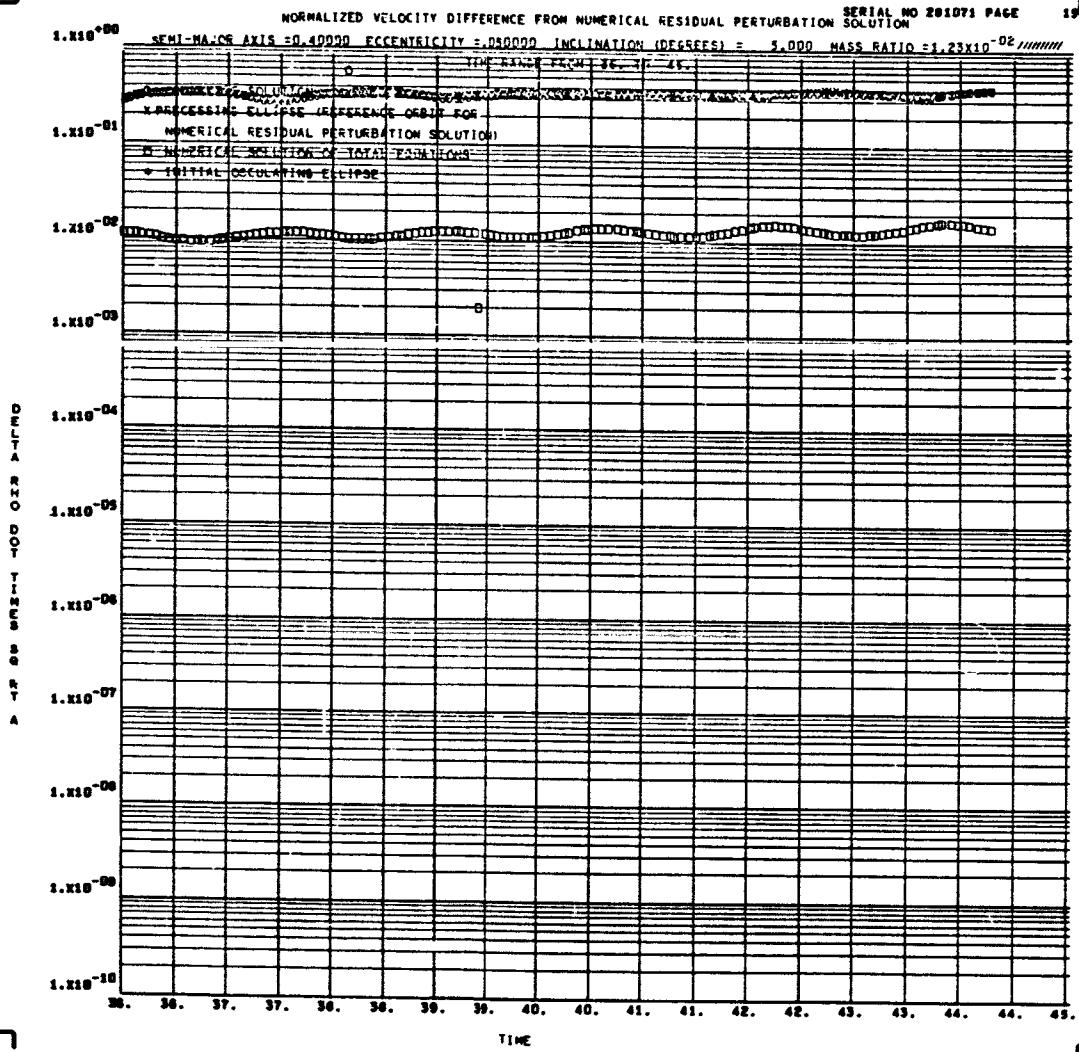
SERIAL NO 281071 PAGE



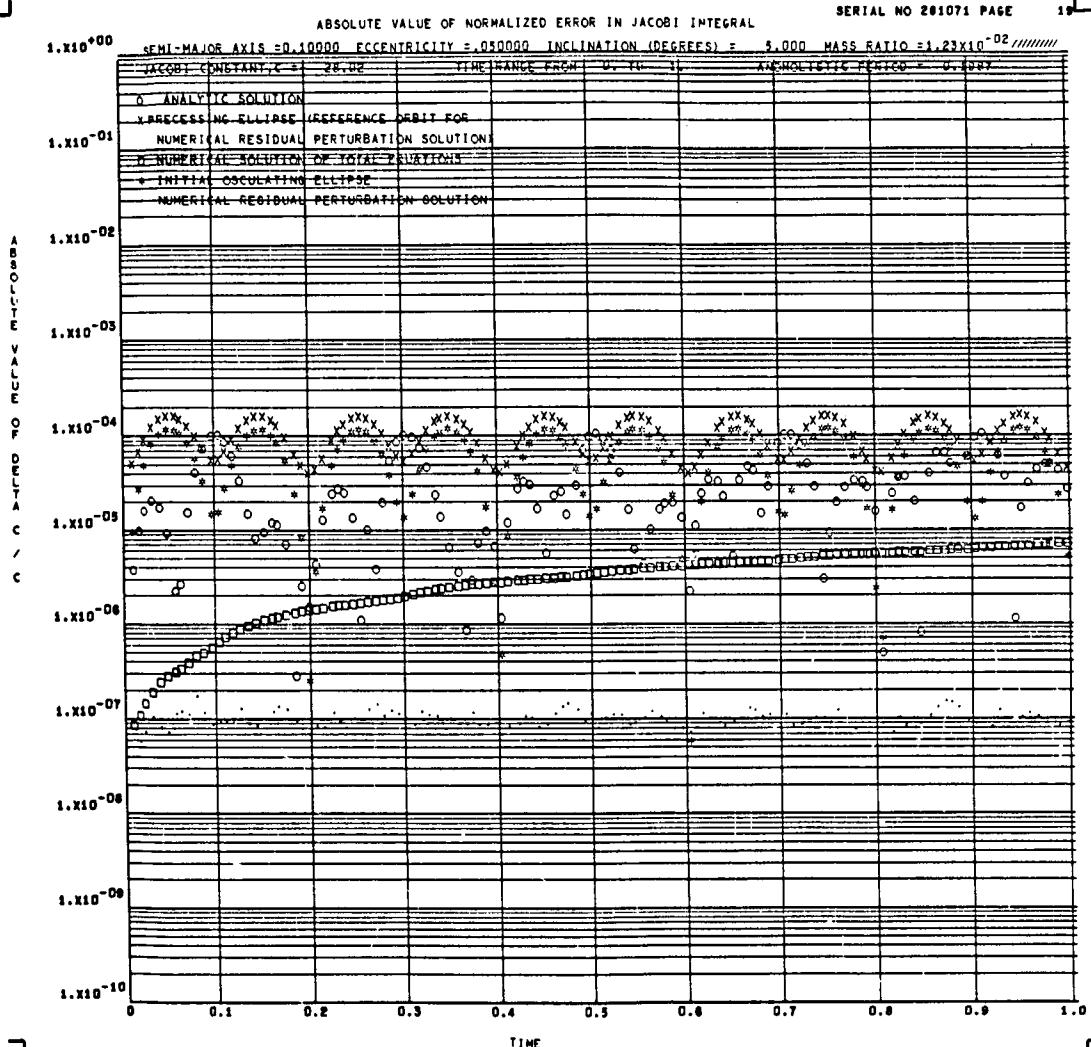
SERIAL NO 281071 PAGE 14



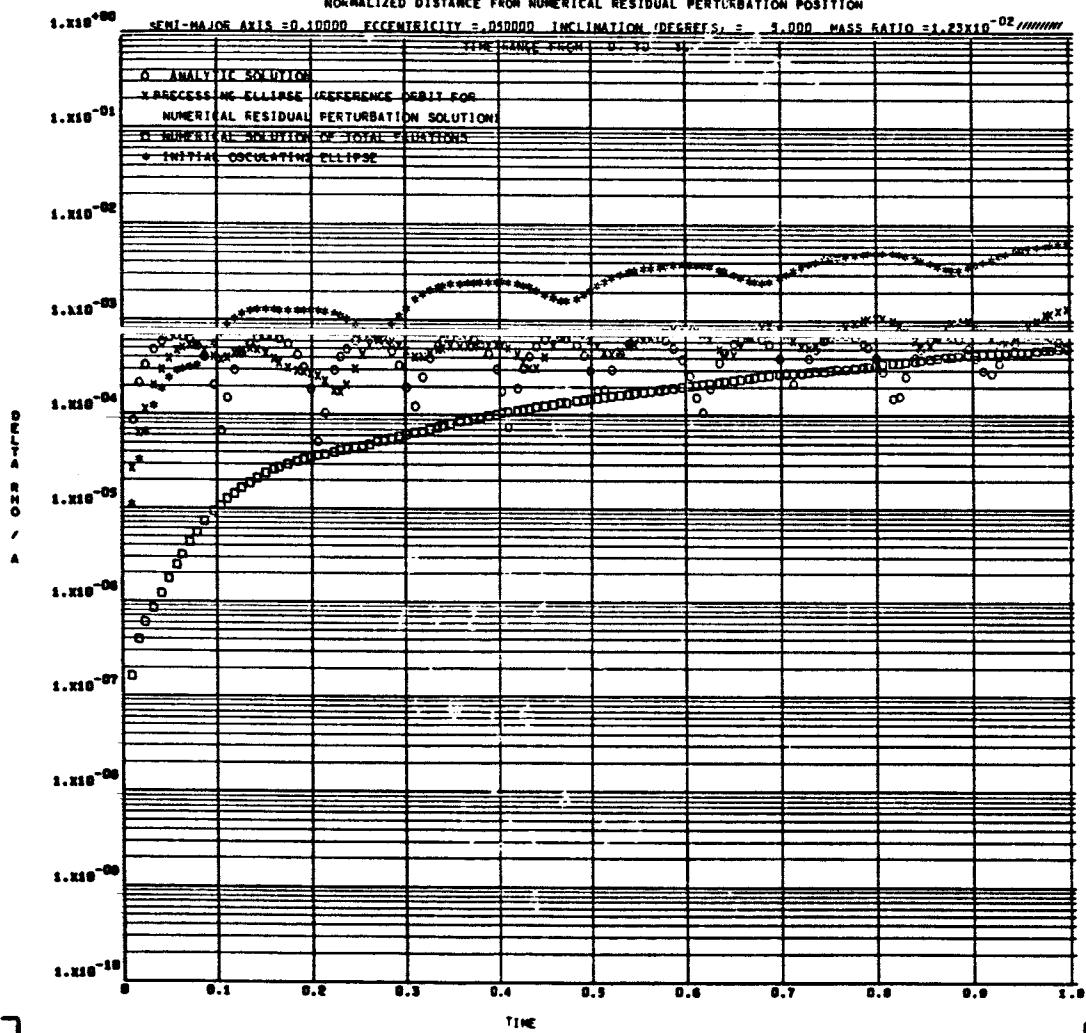
SERIAL NO 281071 PAGE  
SOLUTION



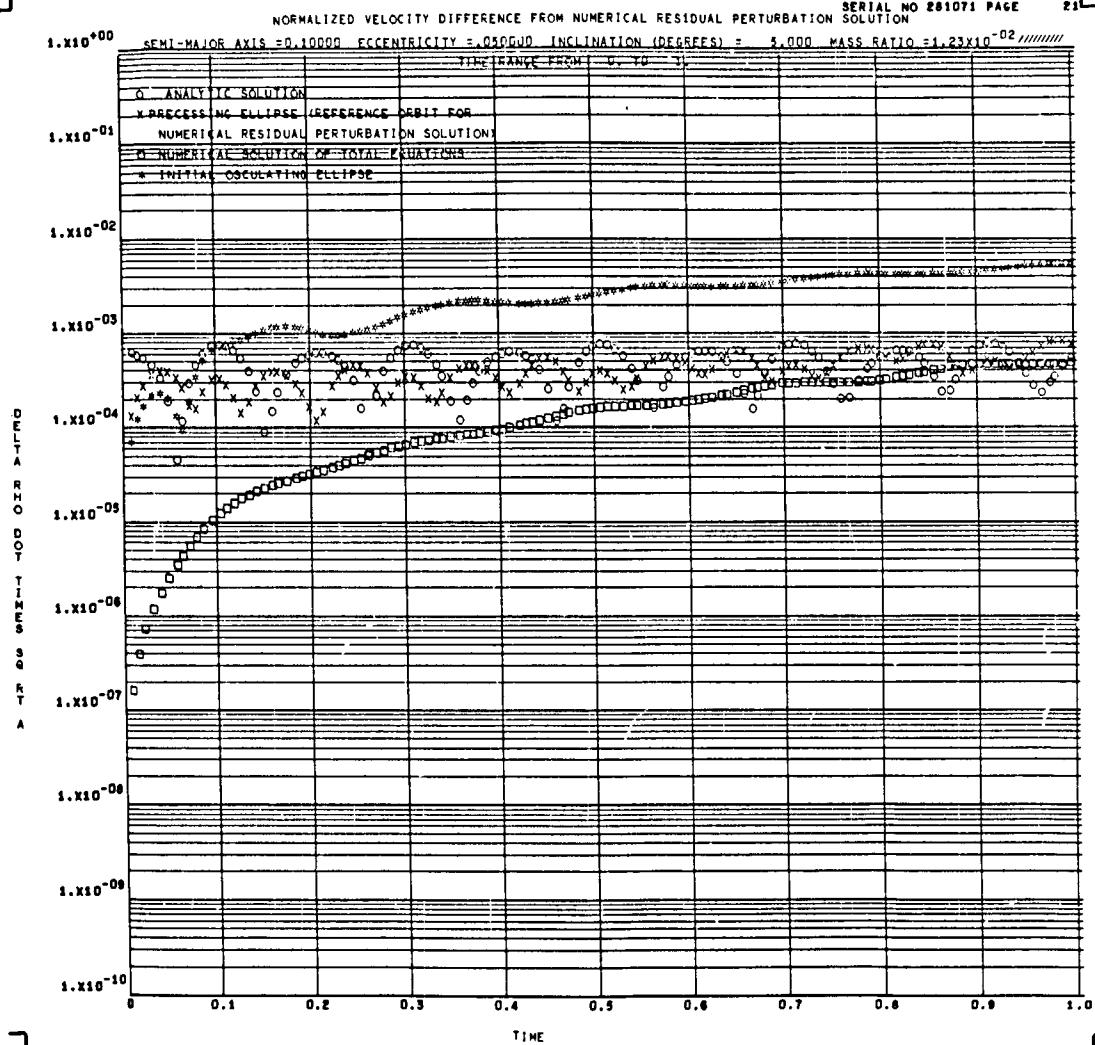
SERIAL NO 281071 PAGE 19

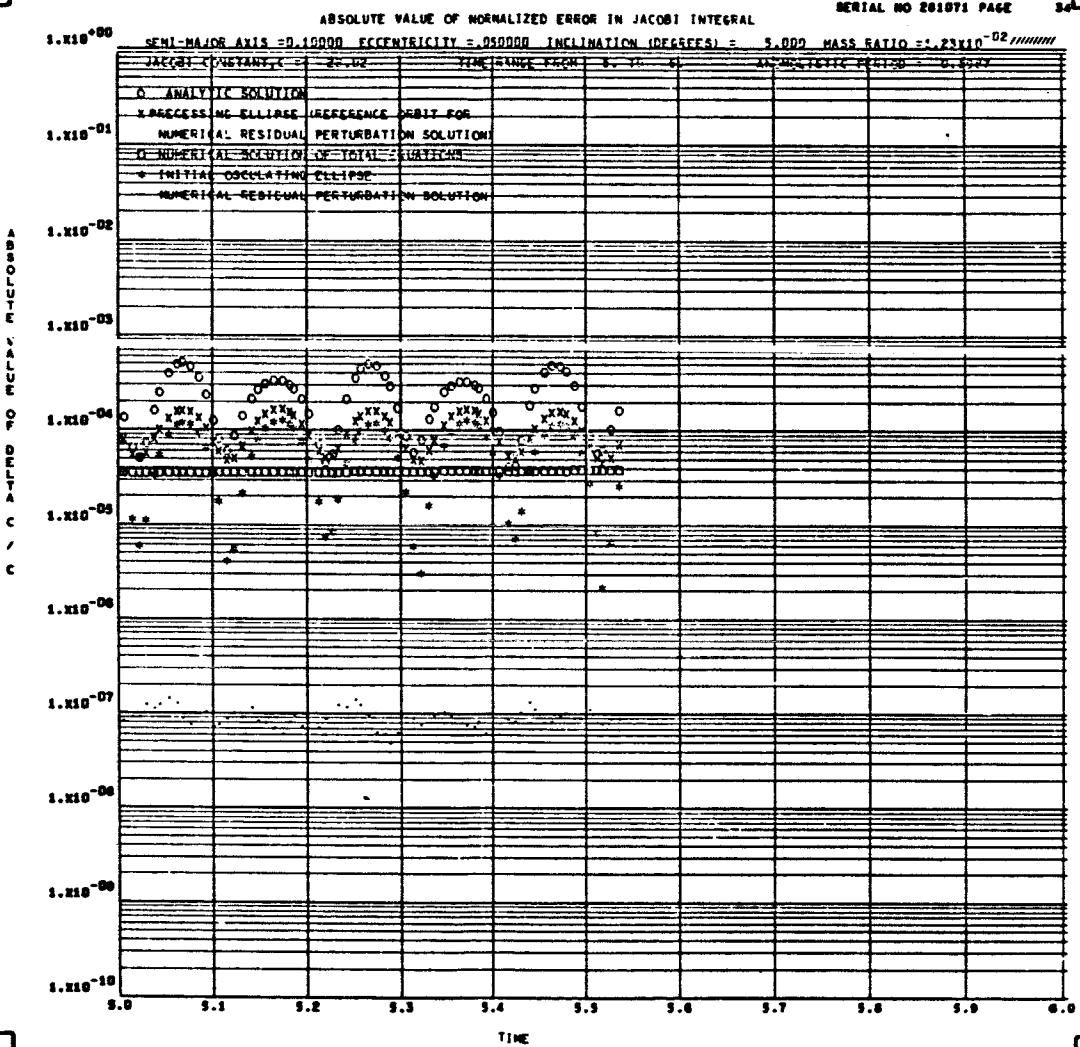


SERIAL NO 281871 PAGE  
TITION

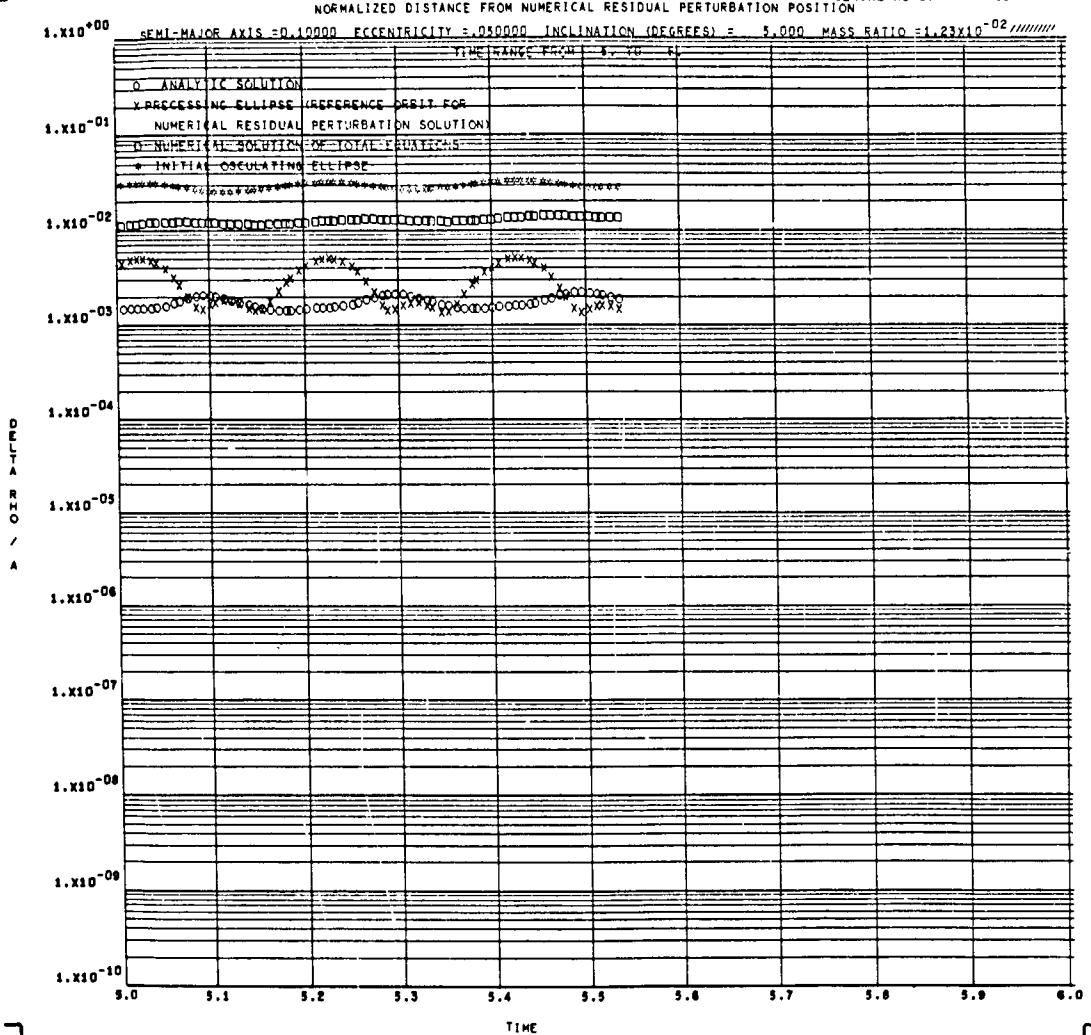


SERIAL NO 281071 PAGE 21

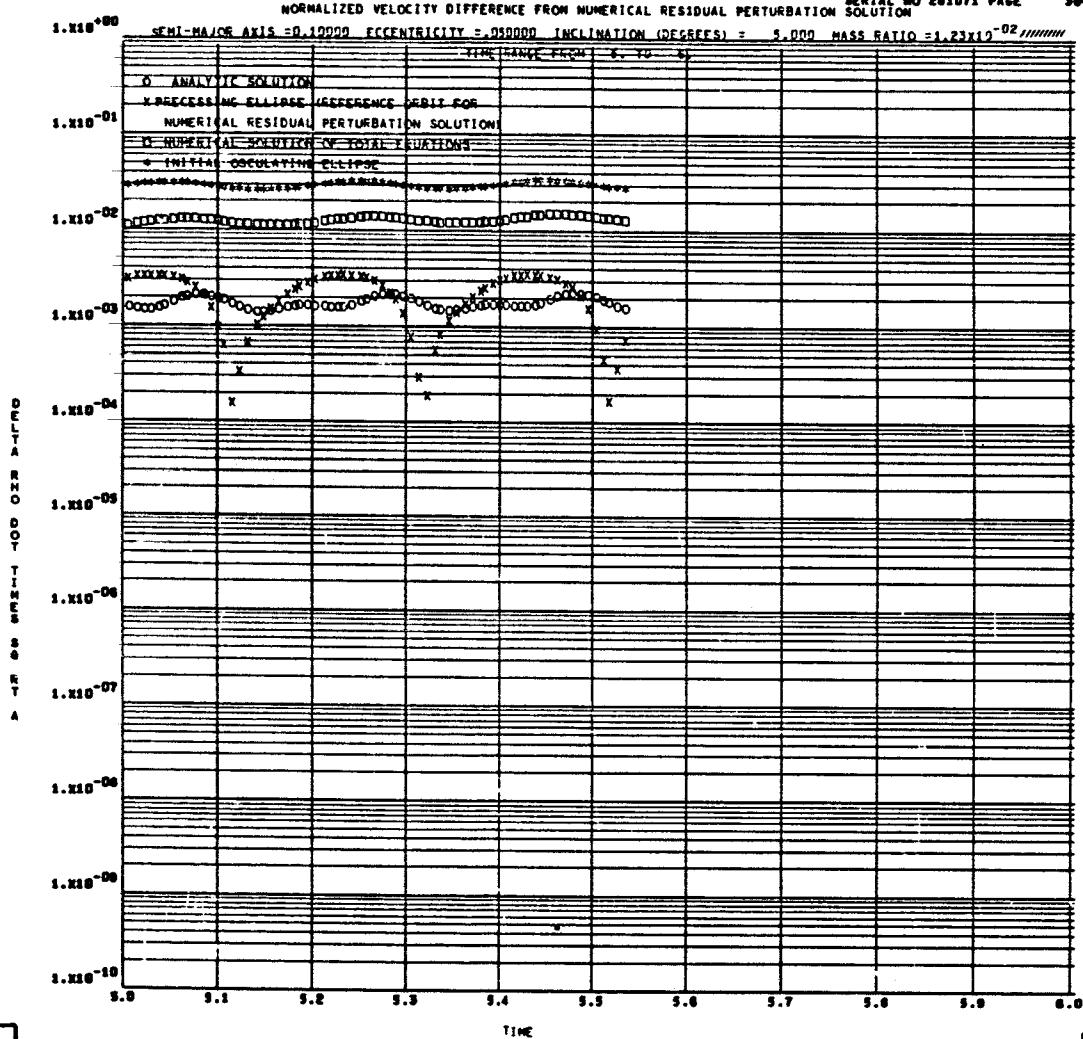


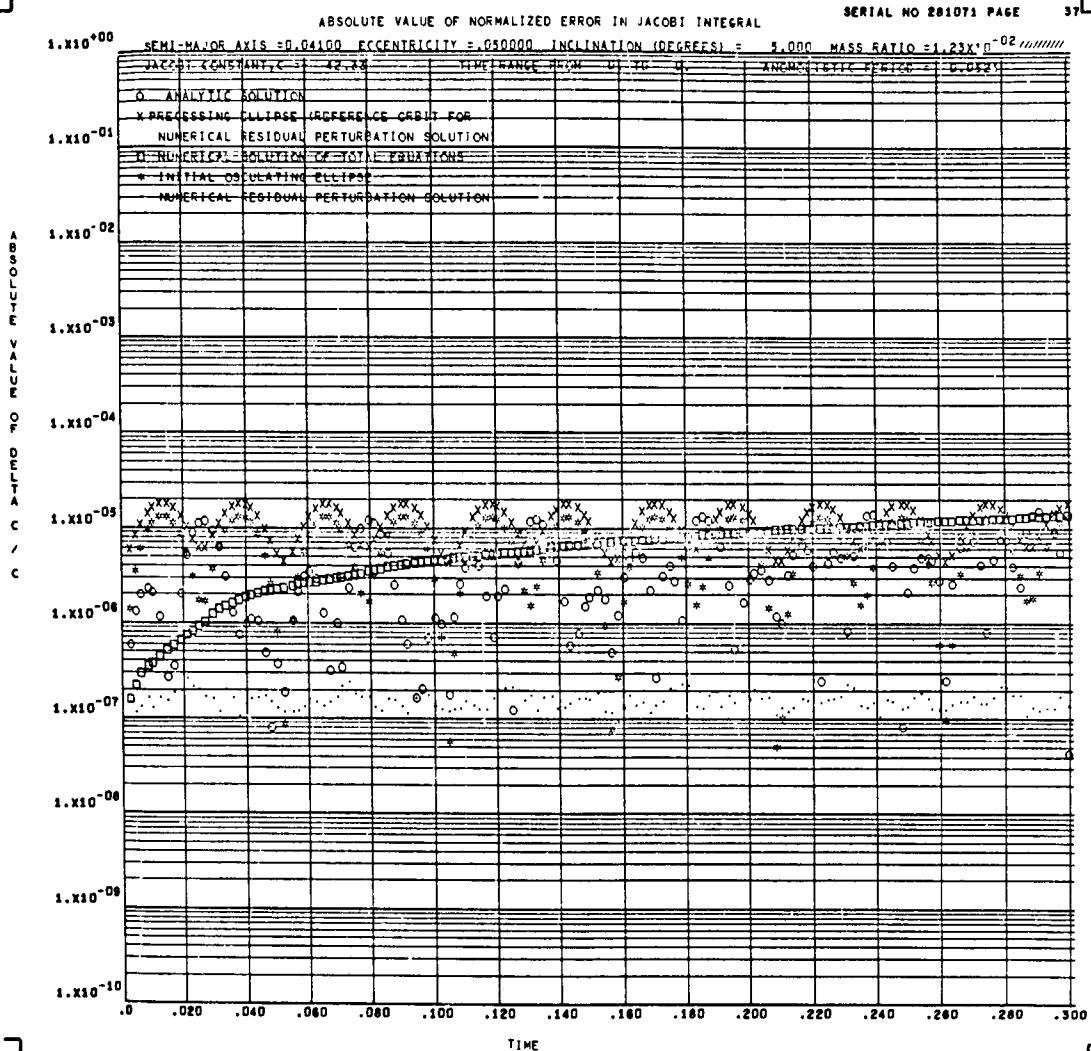


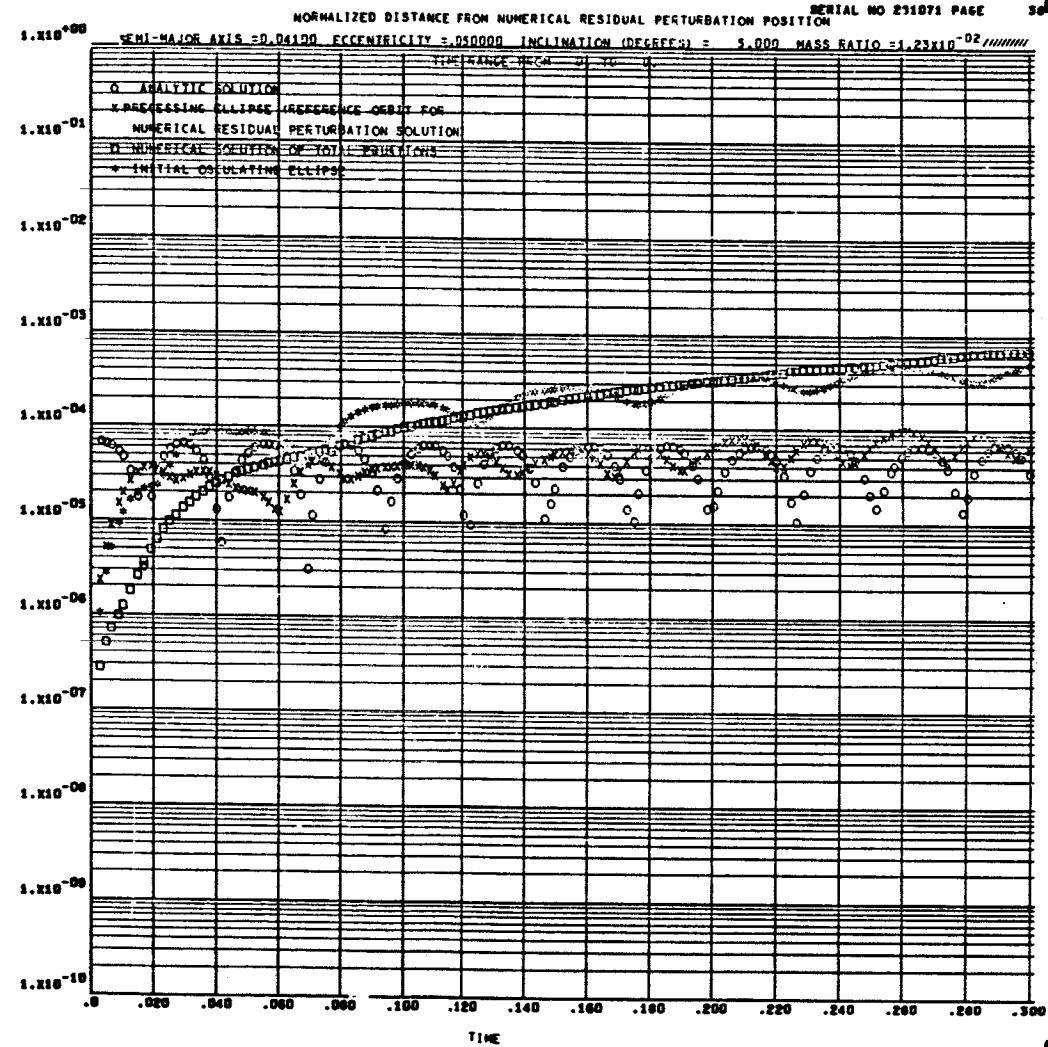
SERIAL NO 201071 PAGE  
1



SERIAL NO 281071 PAGE 30

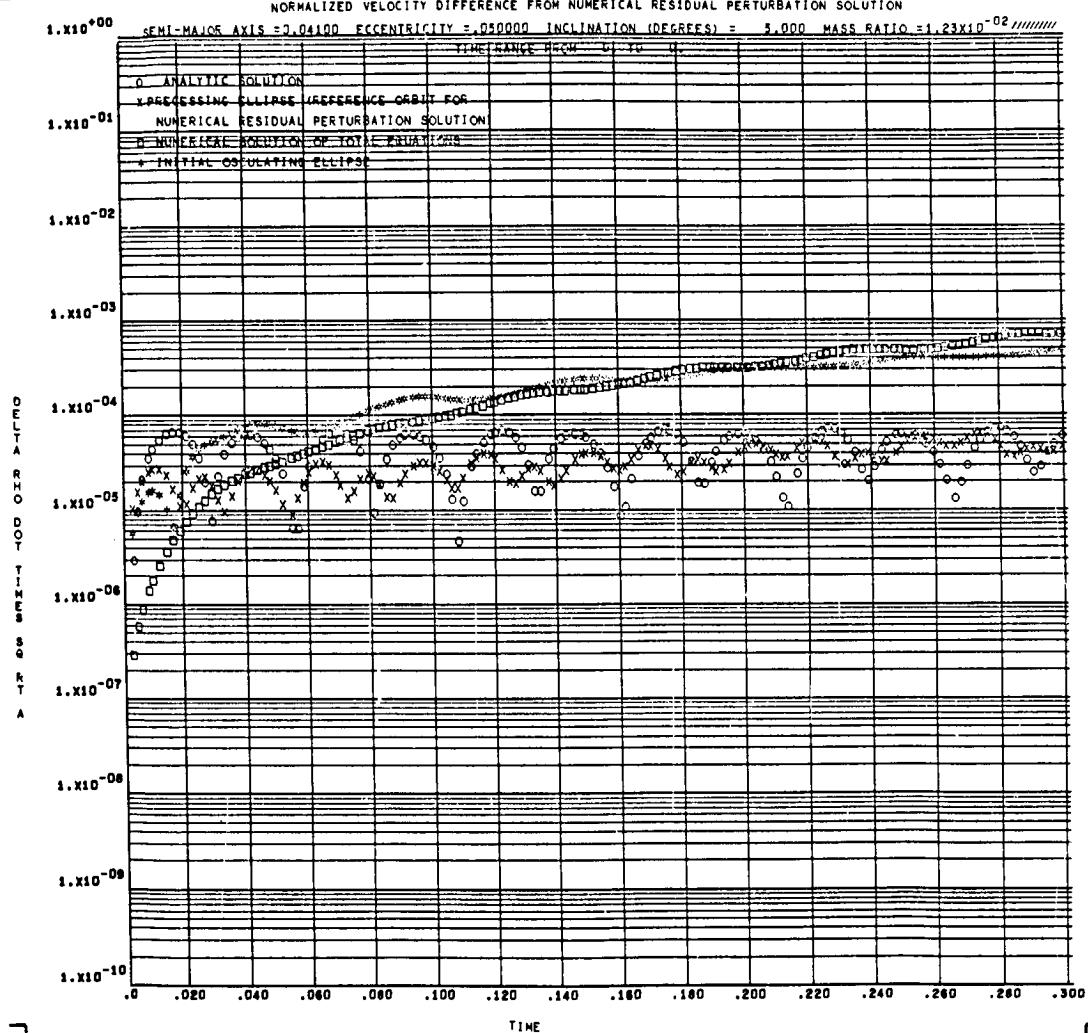






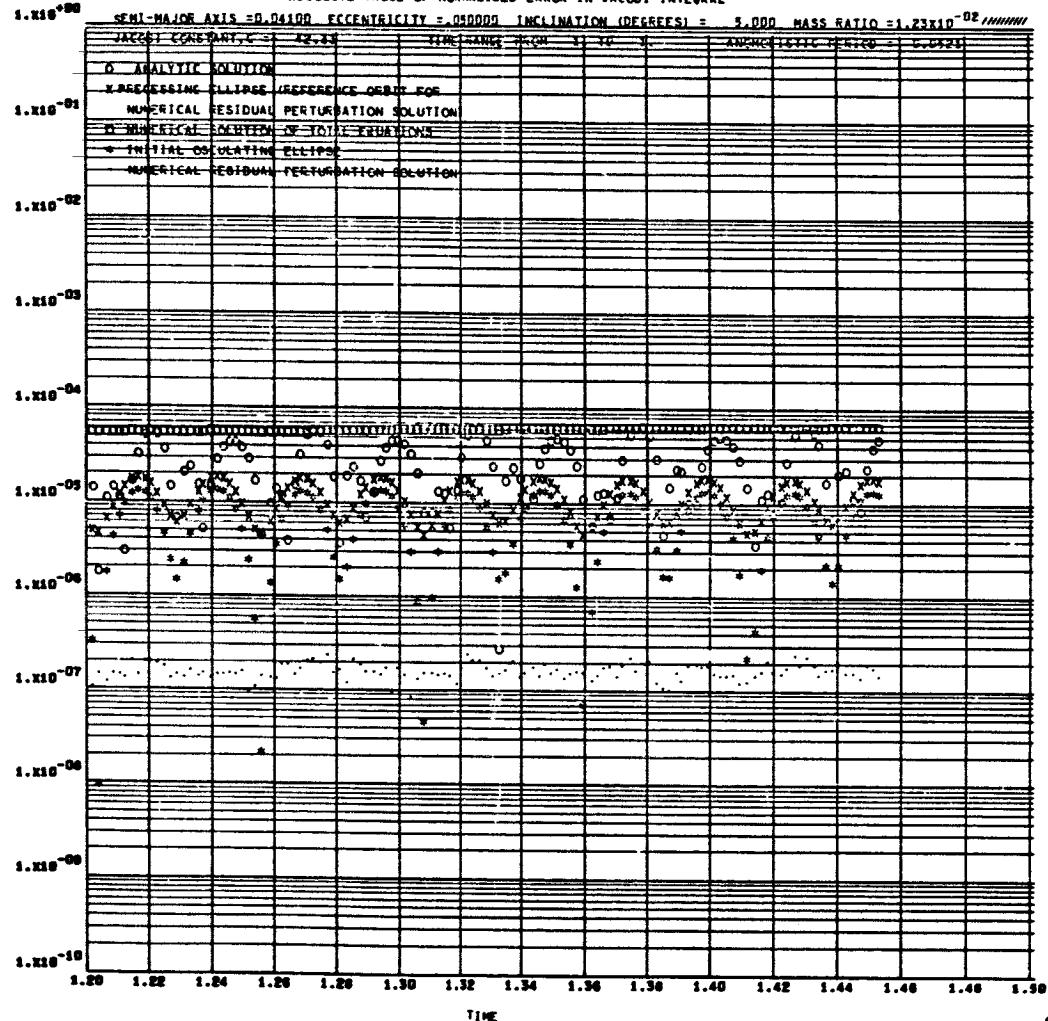
SERIAL NO 281071 PAGE  
SOLUTION

381

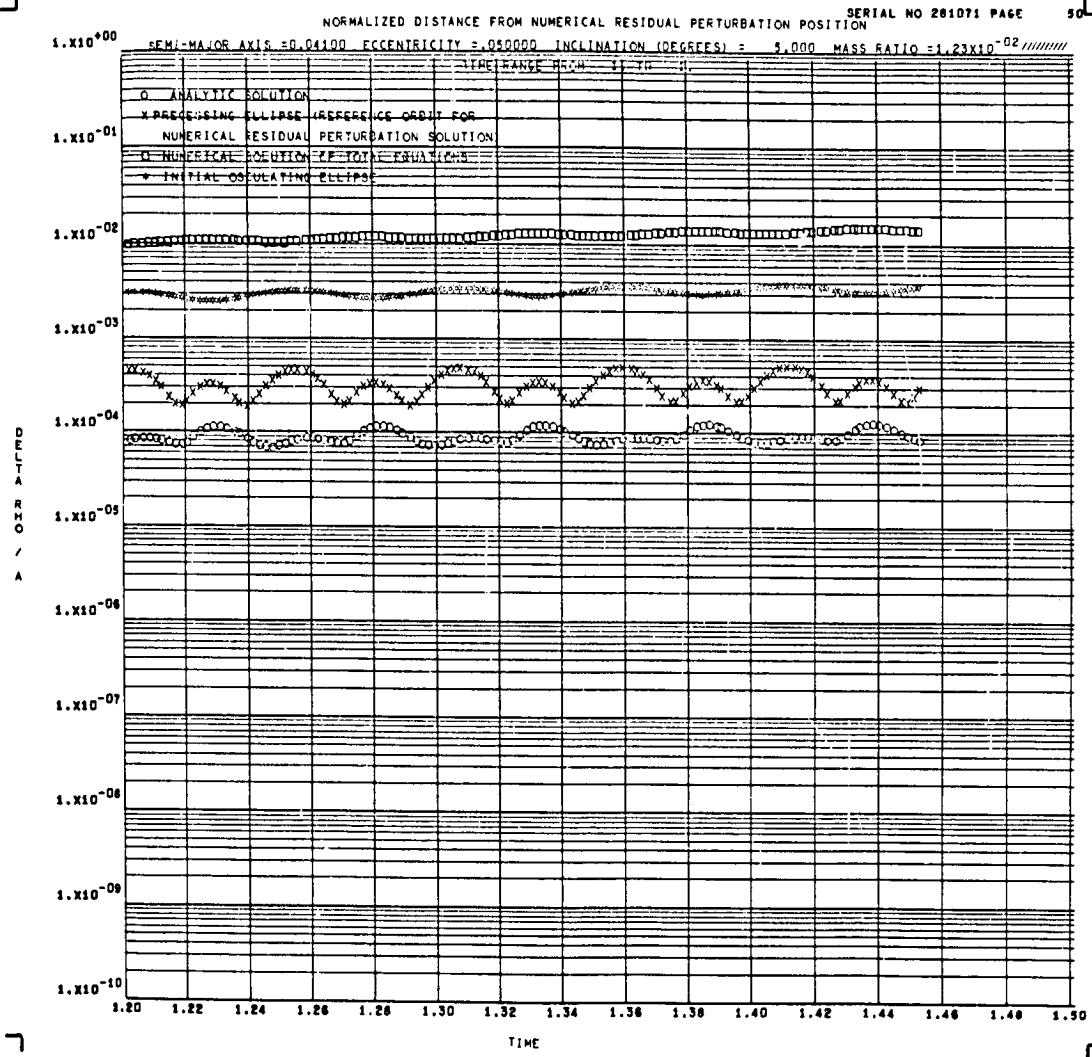


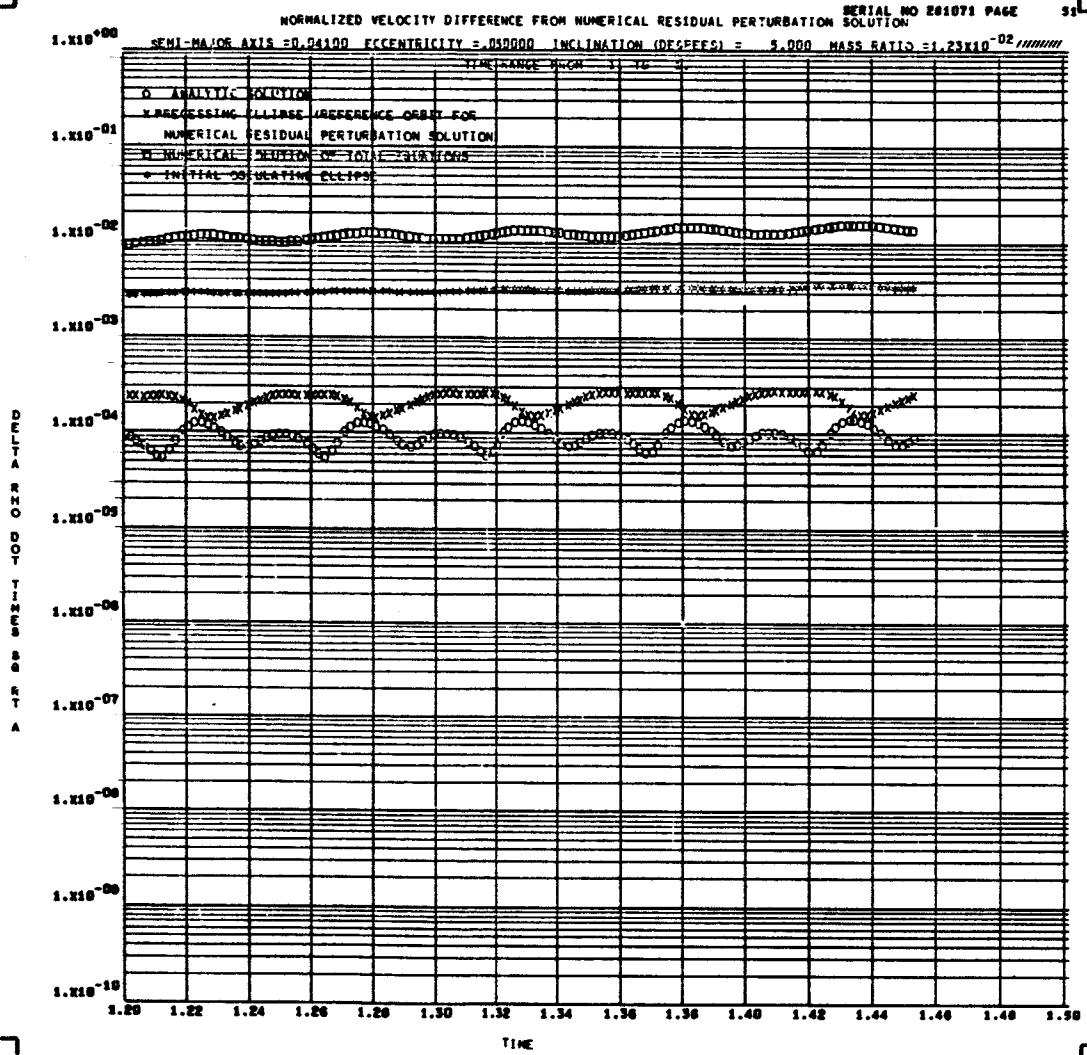
## ABSOLUTE VALUE OF NORMALIZED ERROR IN JACOBI INTEGRAL

SERIAL NO 201071 PAGE 49



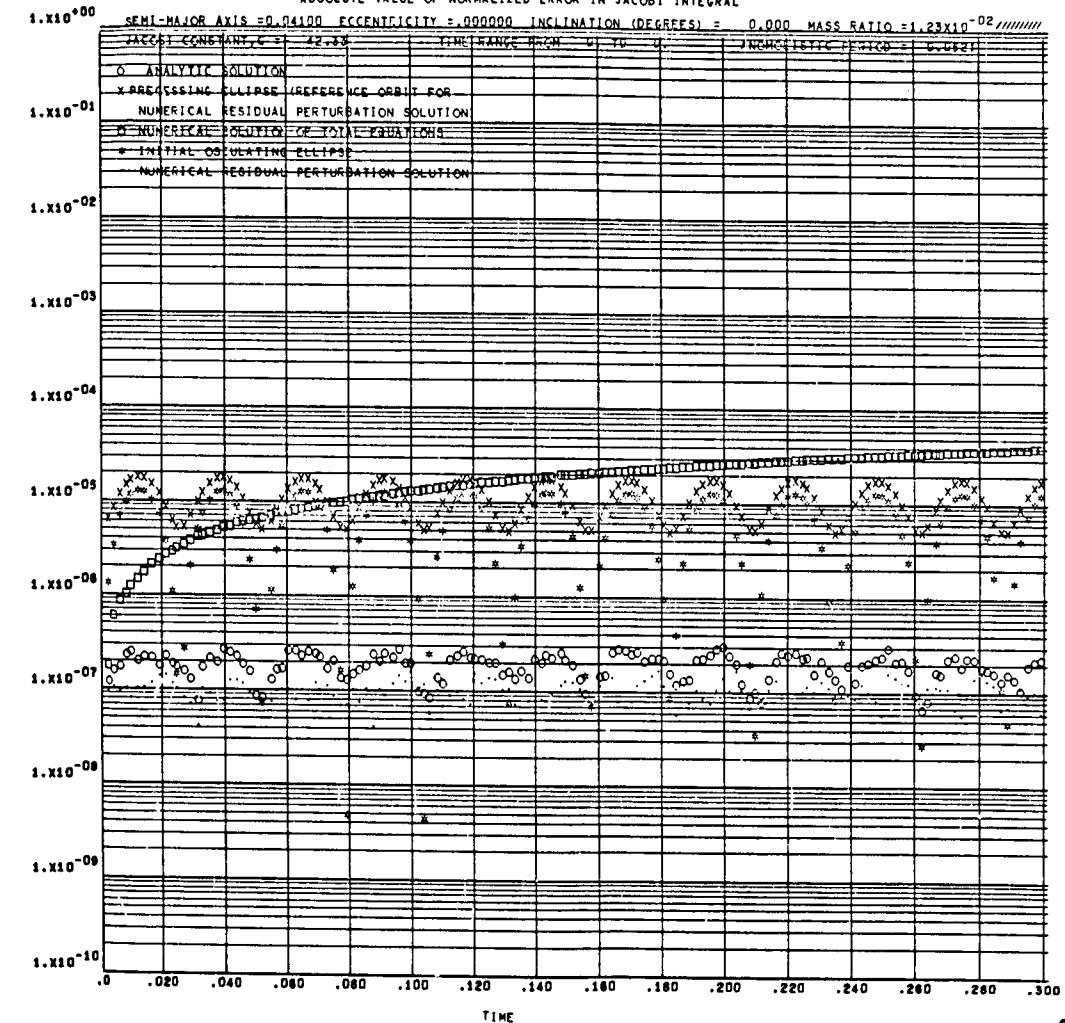
SERIAL NO 281071 PAGE 50

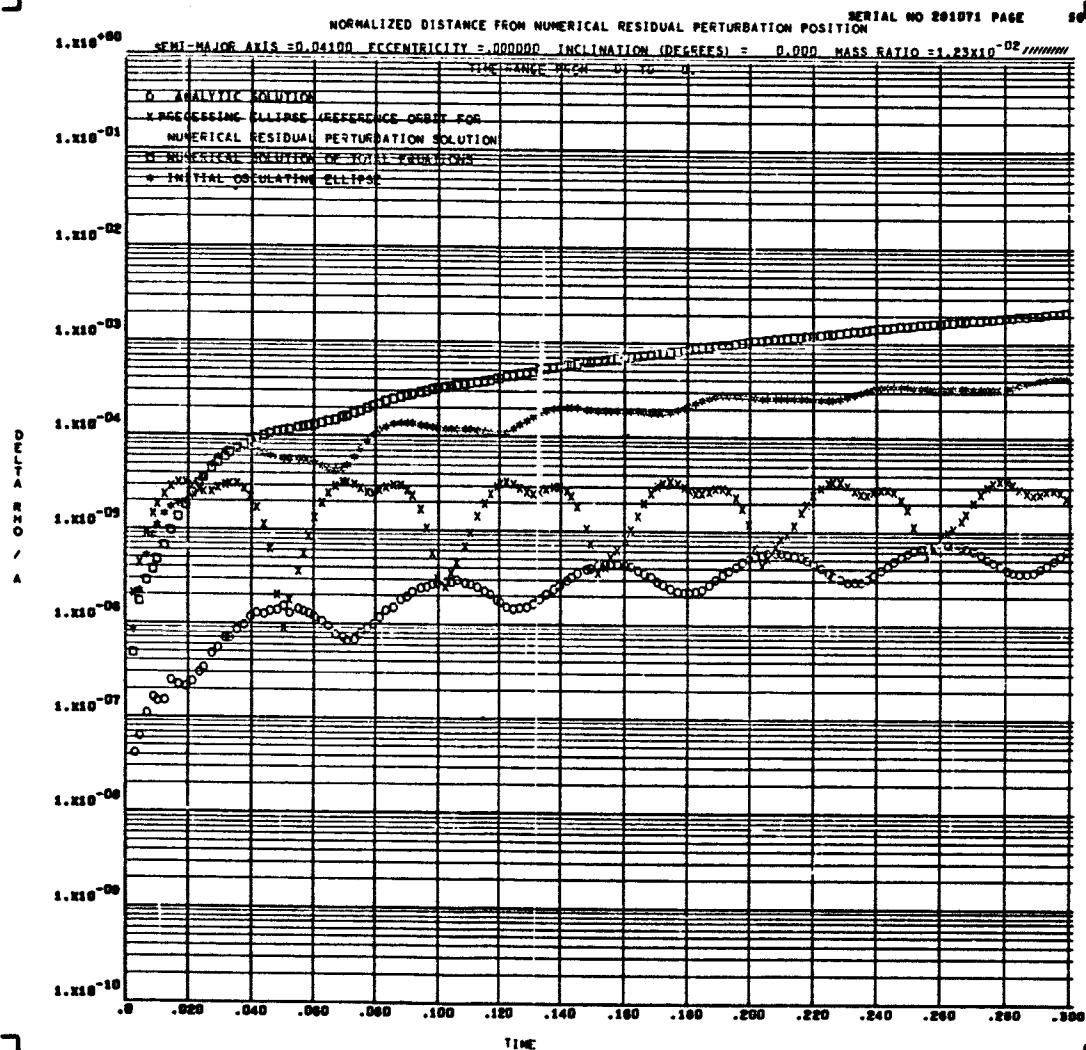


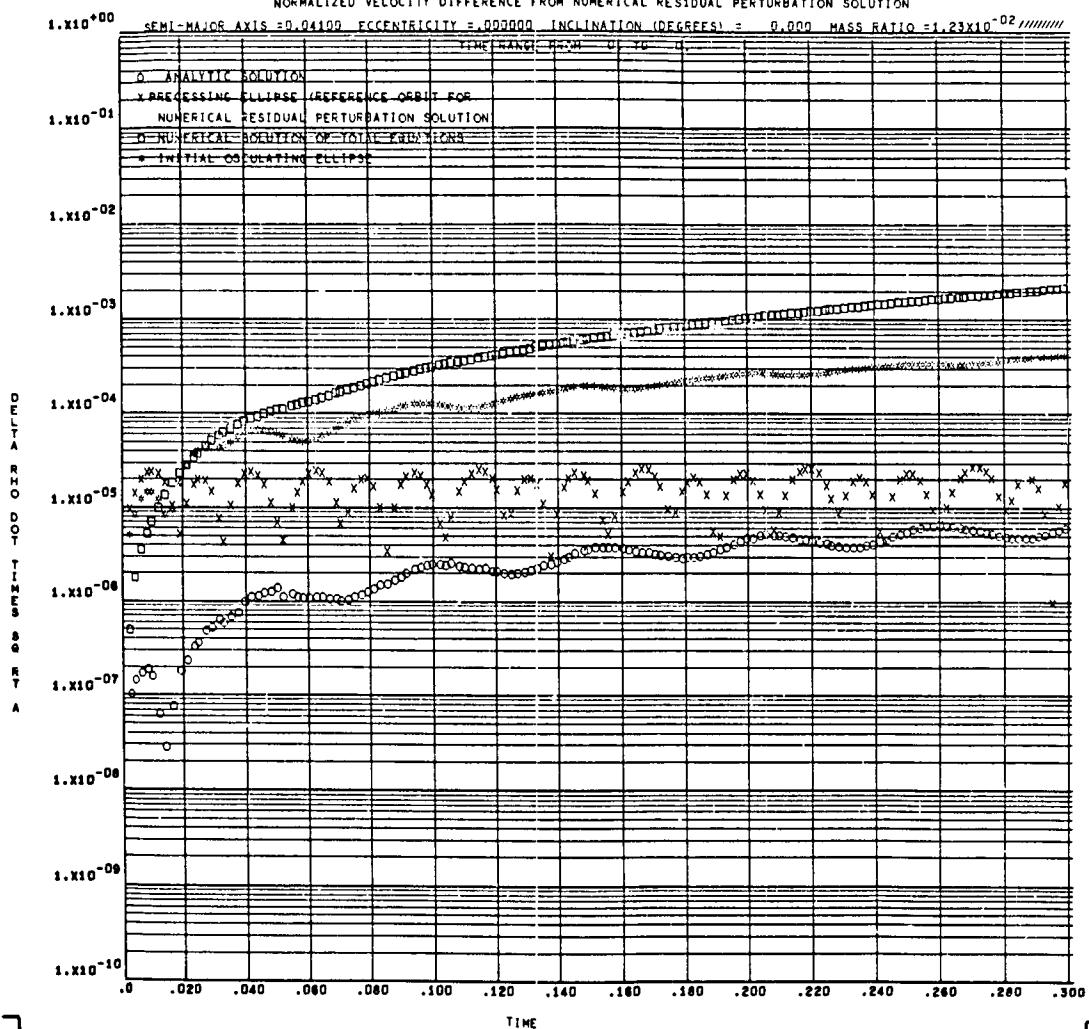


## ABSOLUTE VALUE OF NORMALIZED ERROR IN JACOBI INTEGRAL

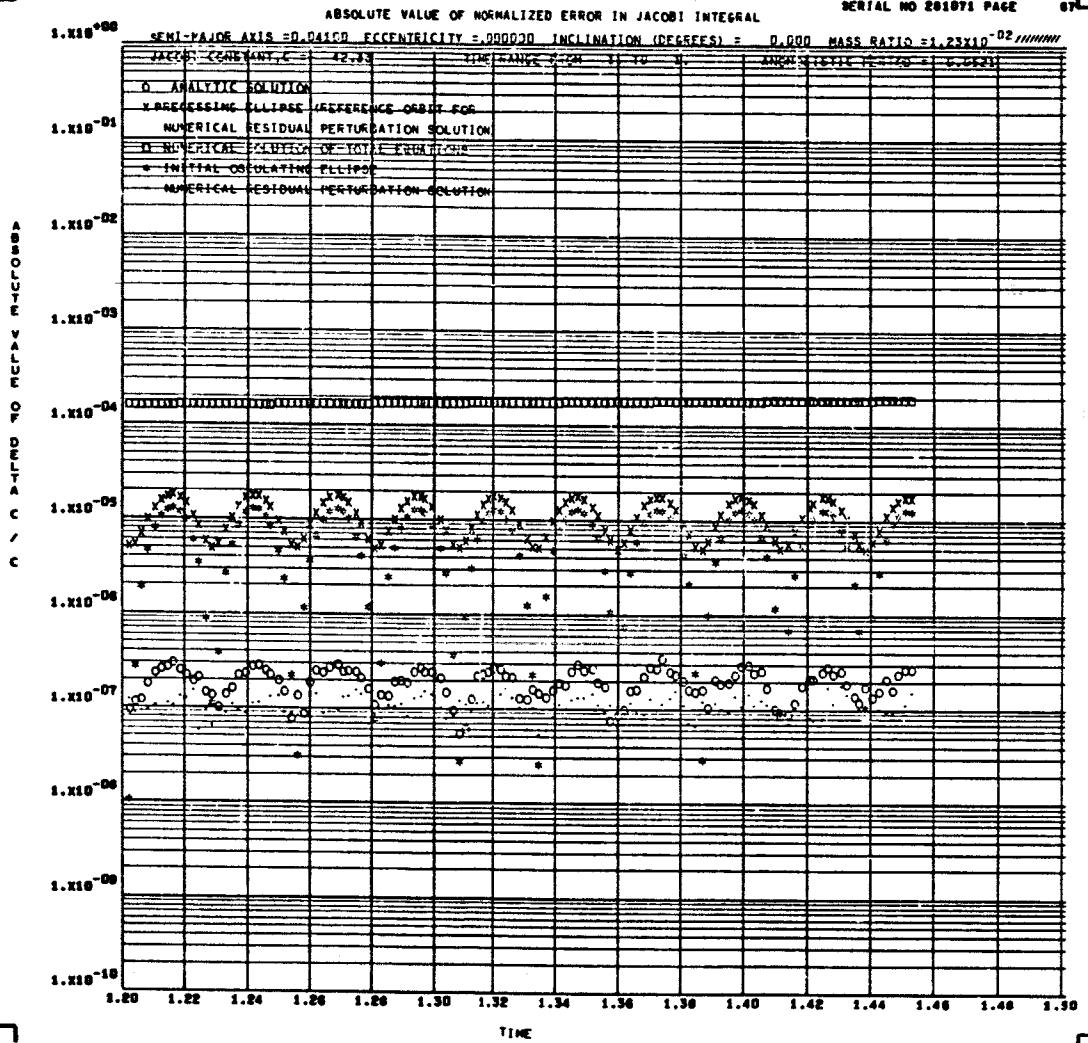
SERIAL NO 281071 PAGE 59



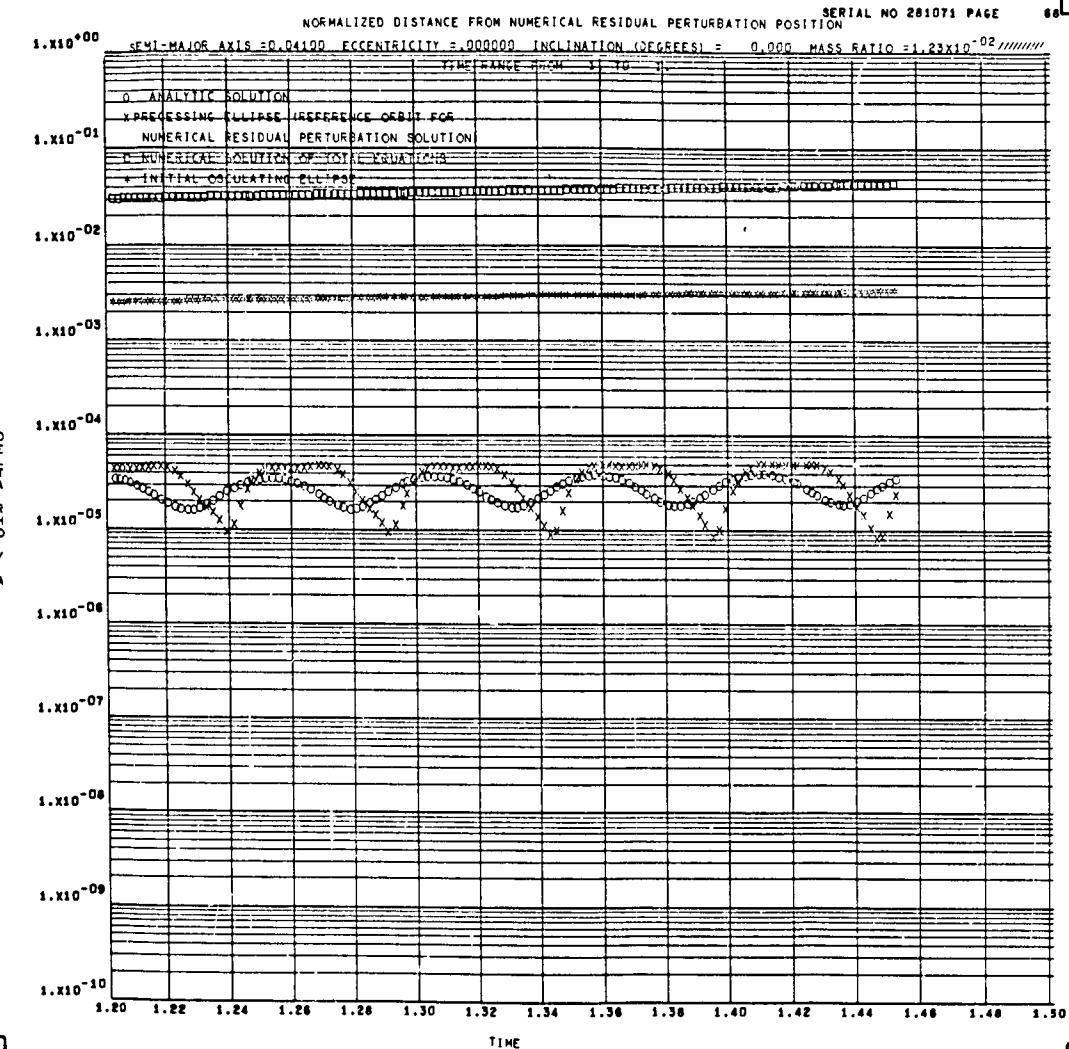


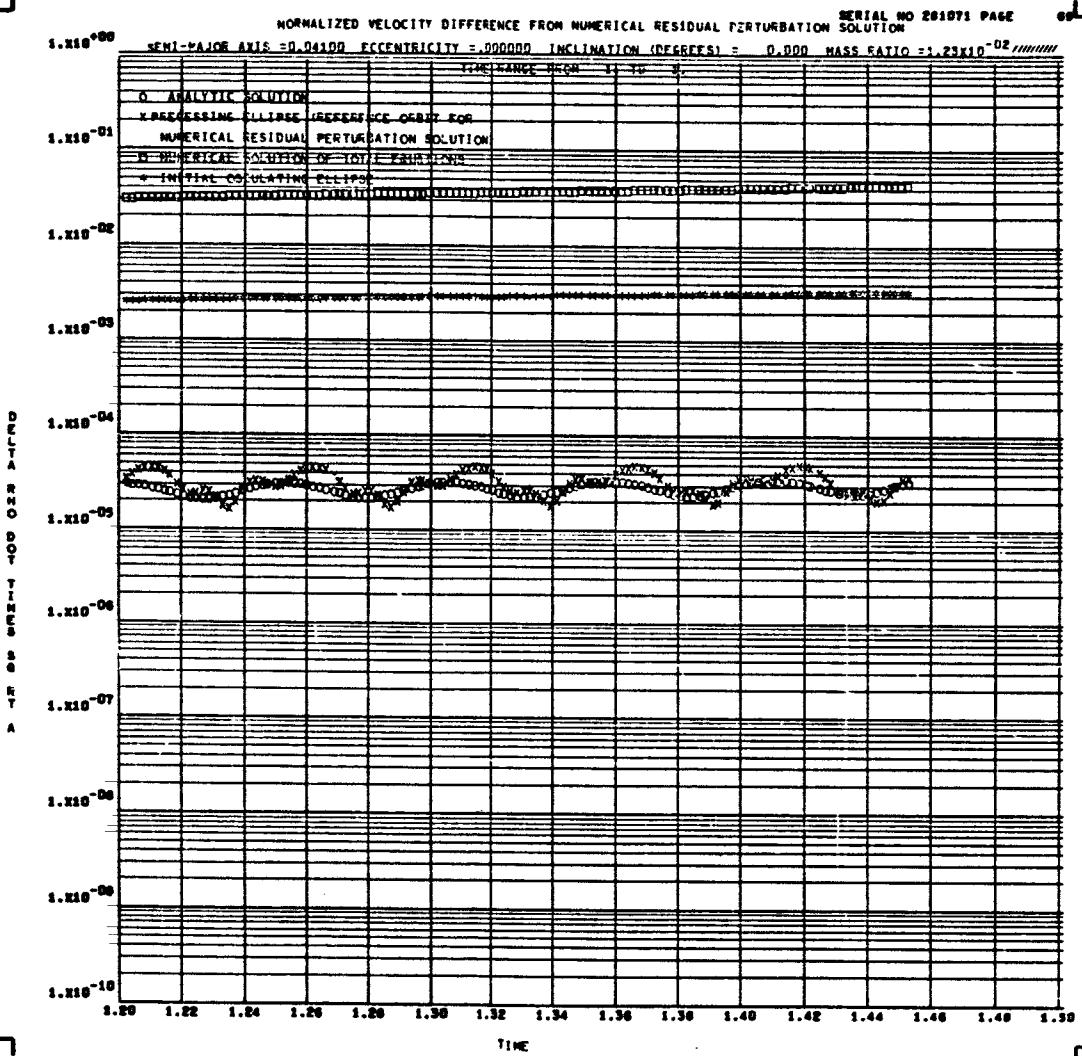


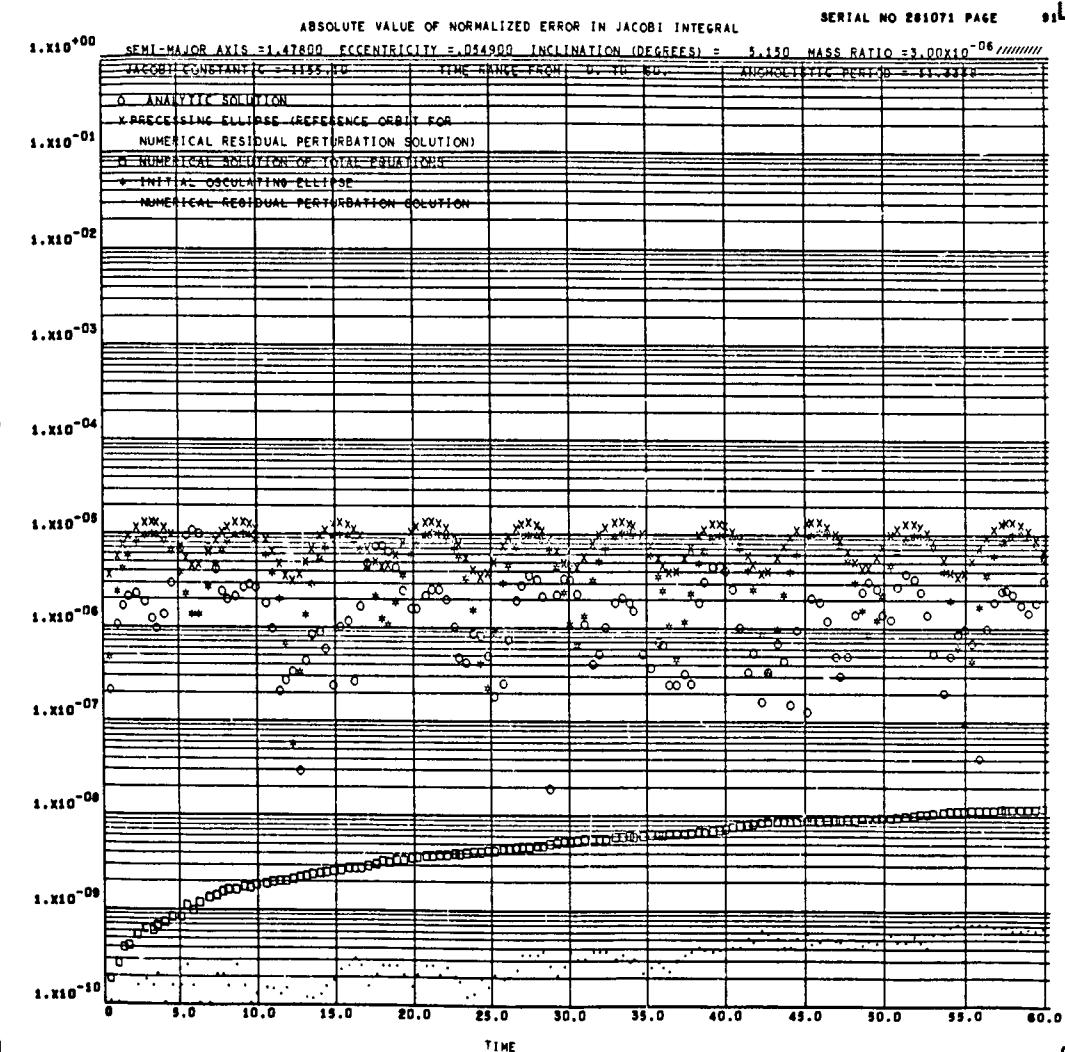
SERIAL NO 281871 PAGE



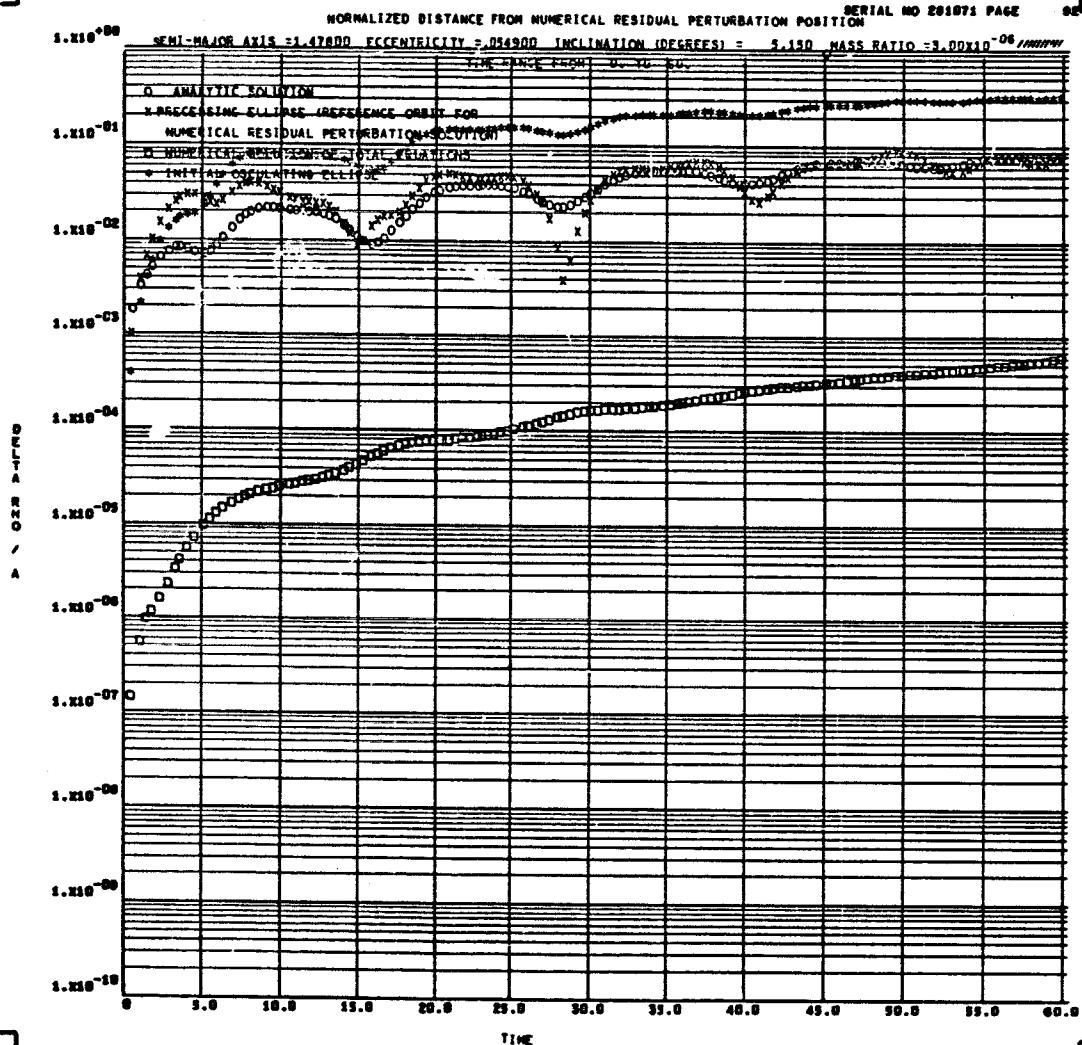
SERIAL NO 281071 PAGE 88

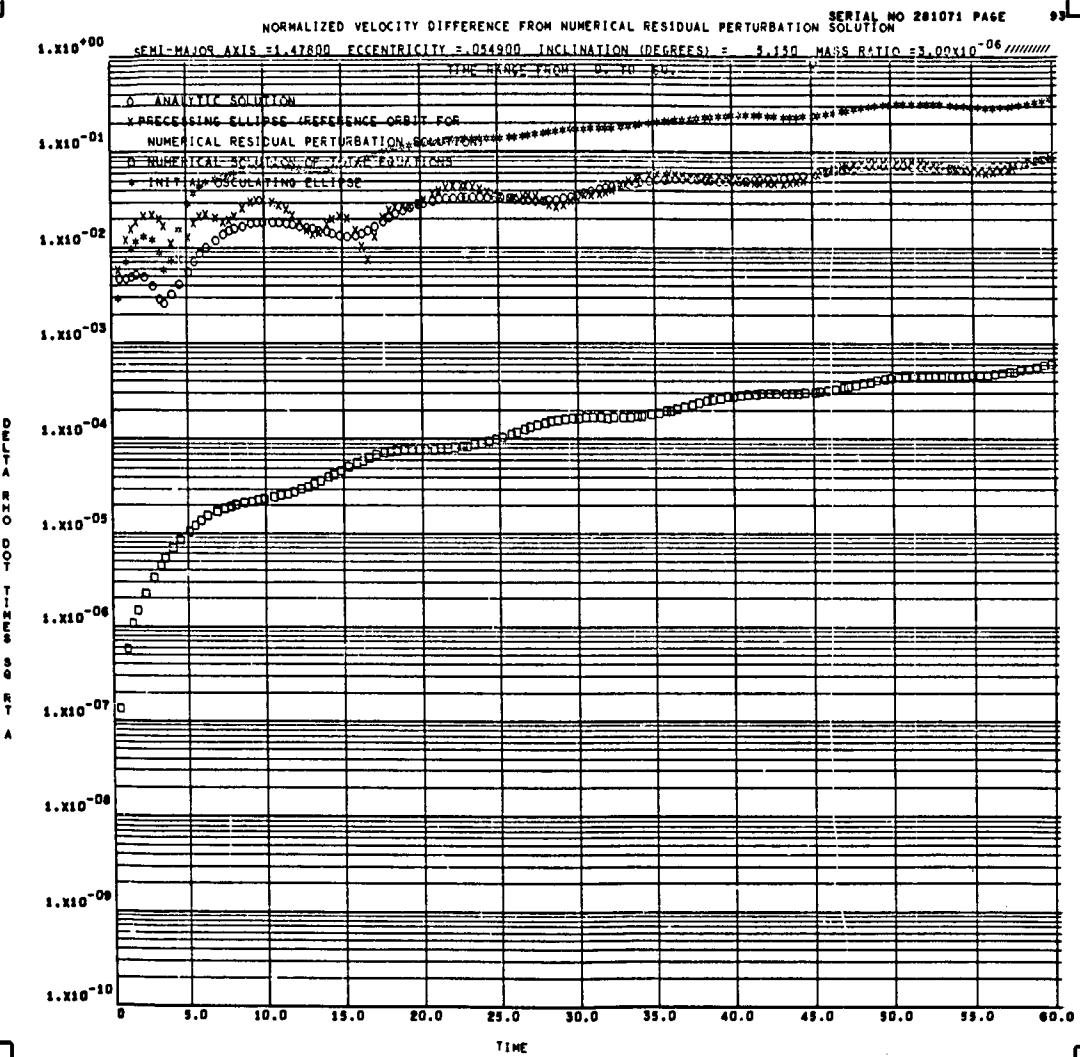




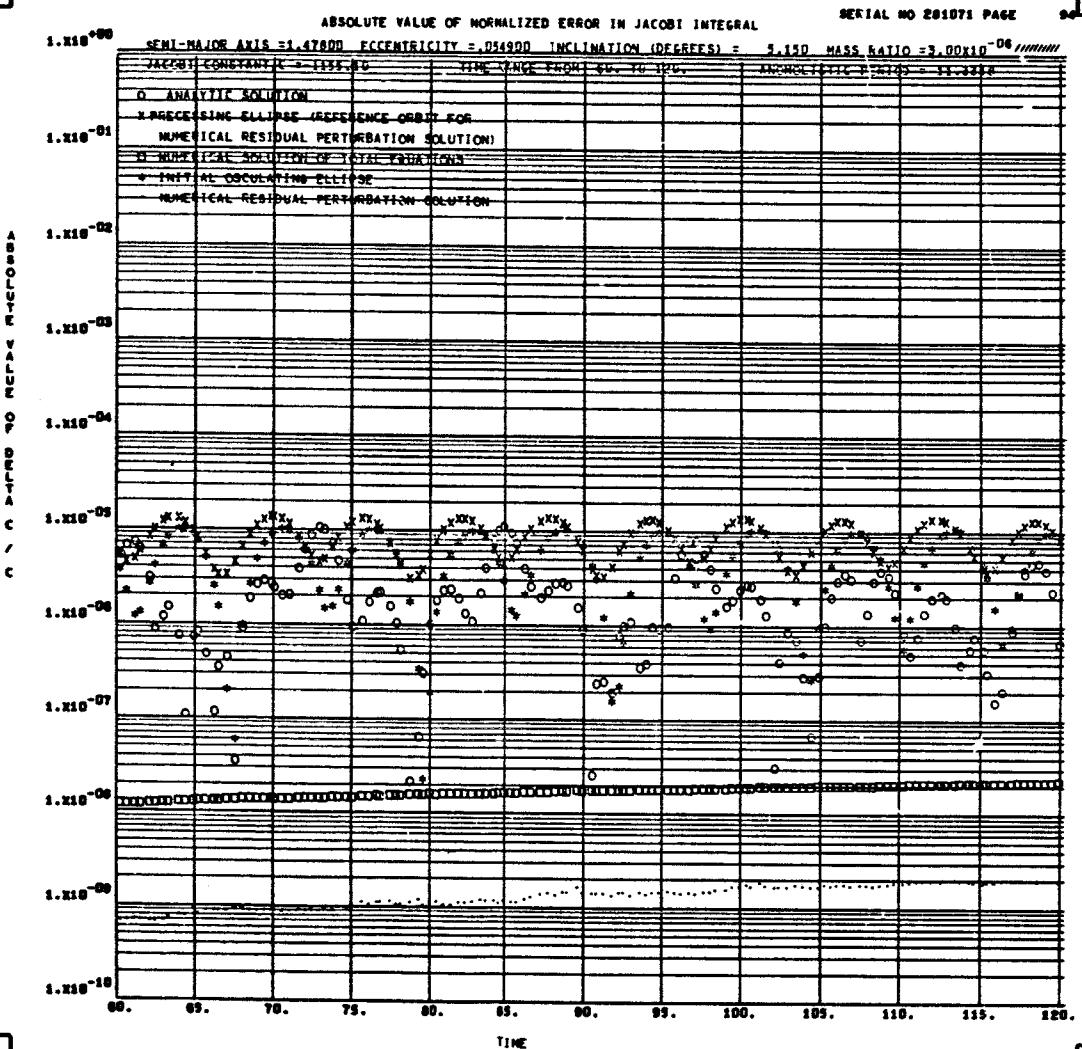


SERIAL NO 201871 PAGE 02

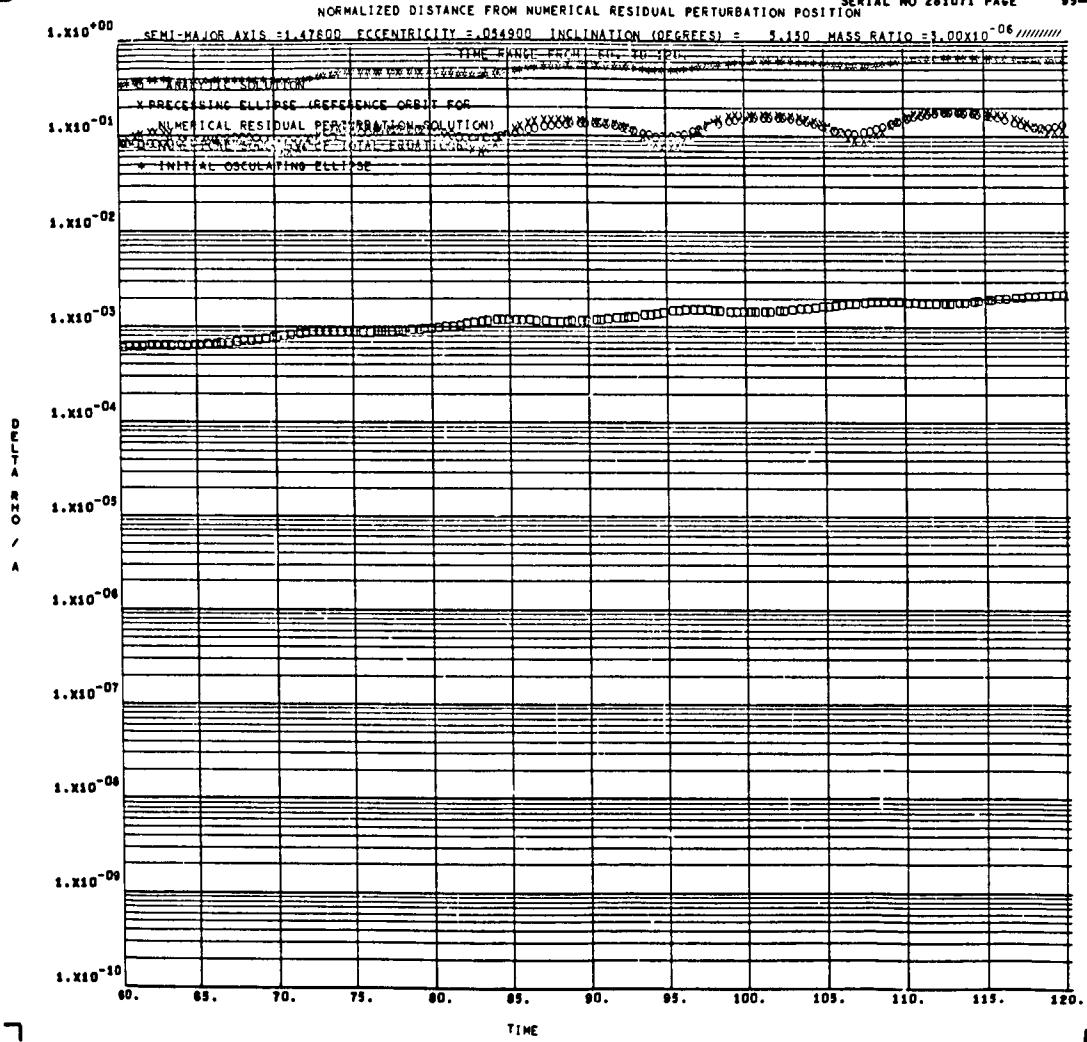




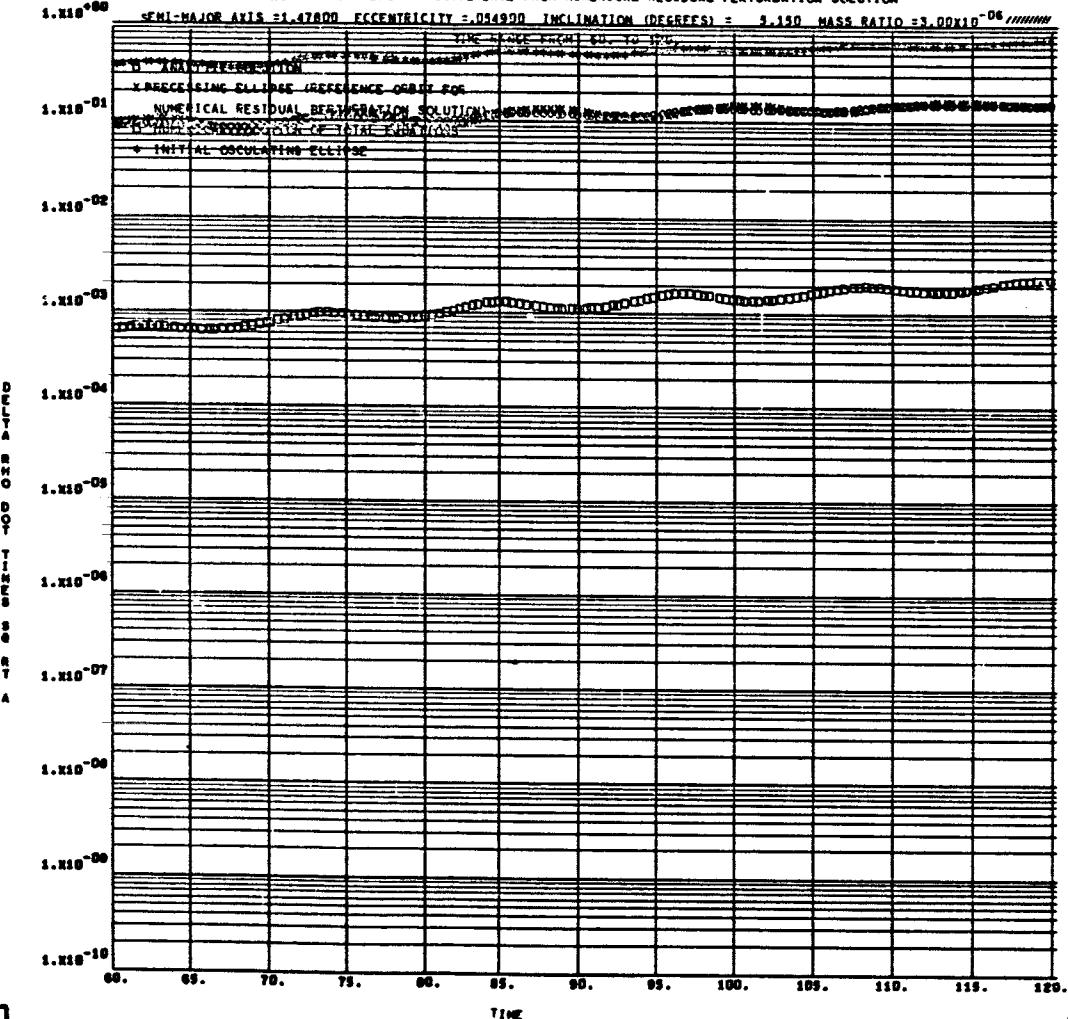
SERIAL NO 281071 PAGE 94

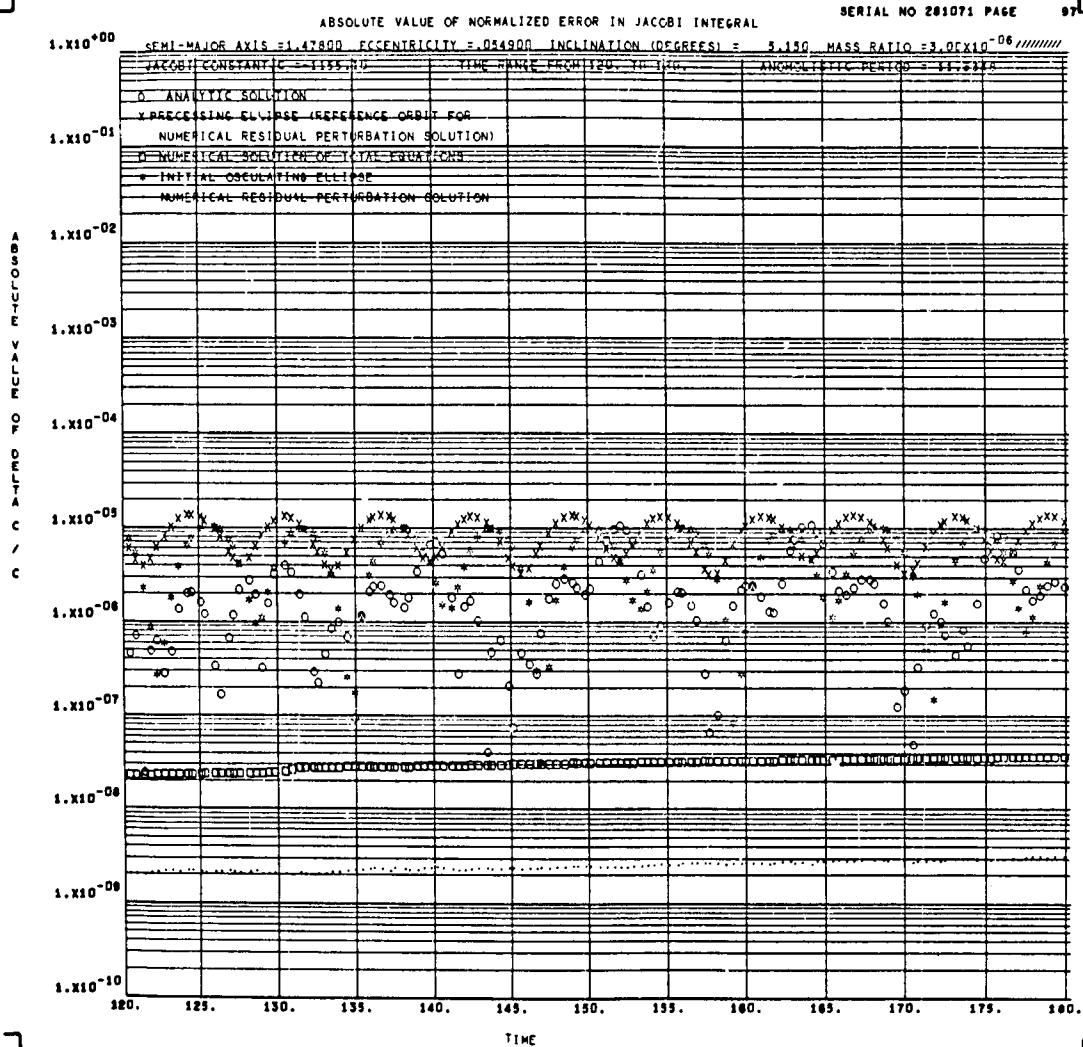


SERIAL NO 281071 PAGE 89

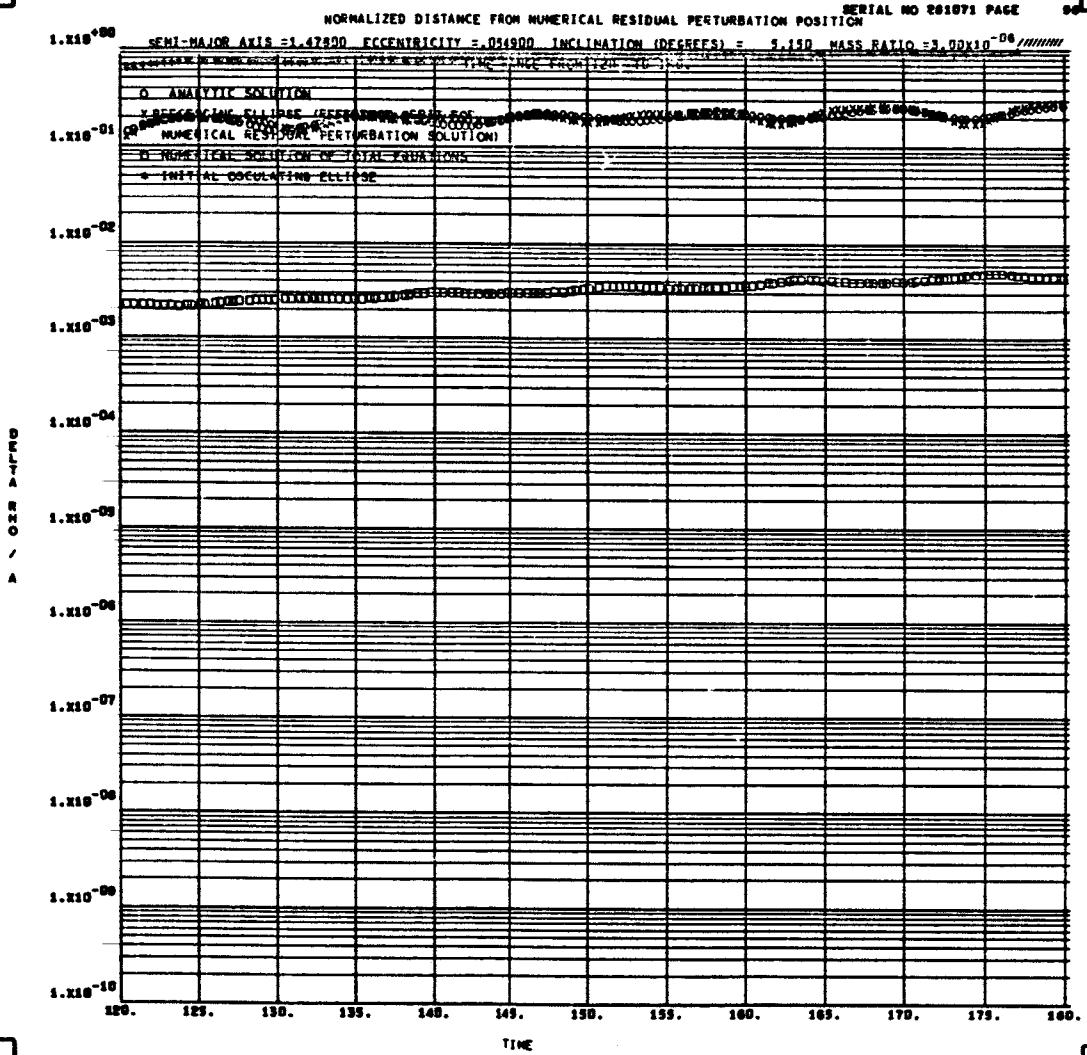


NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION

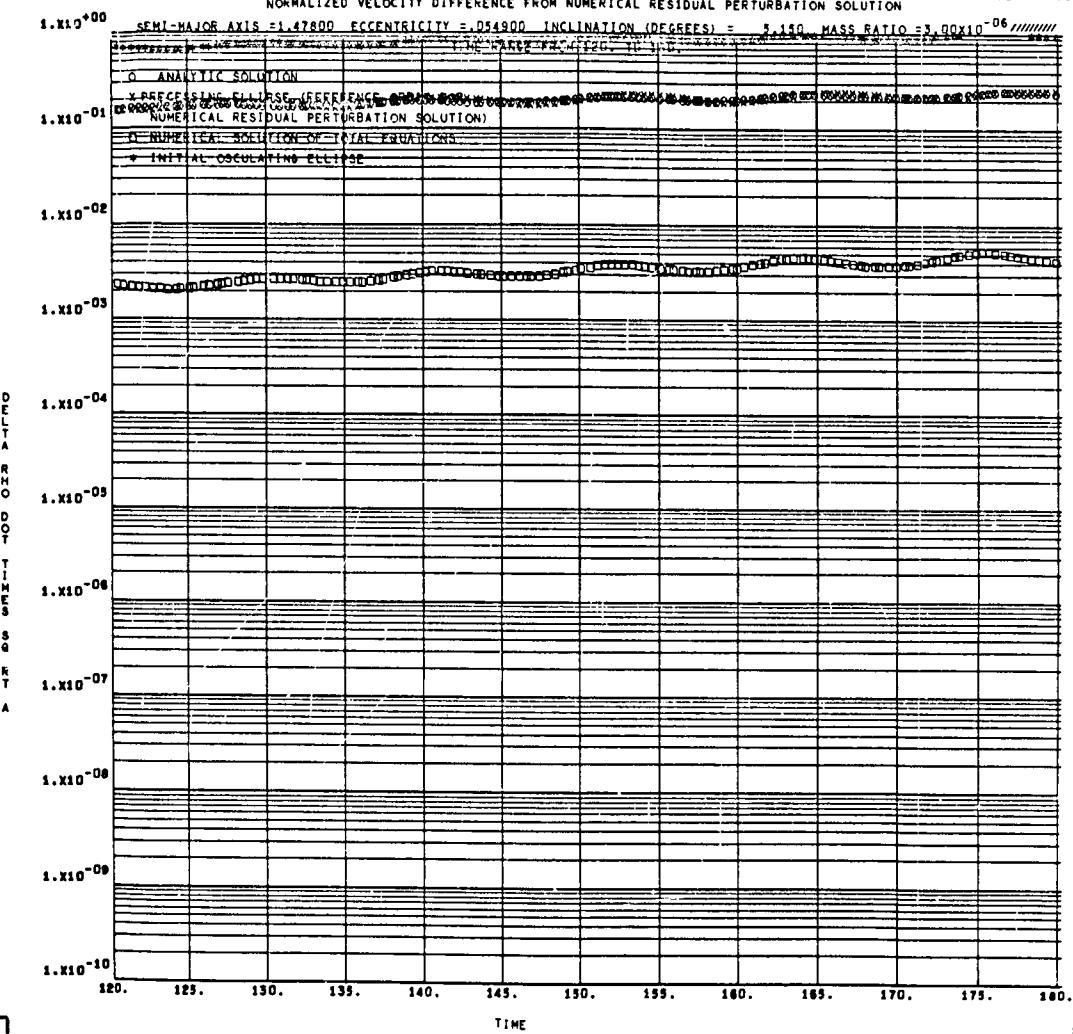


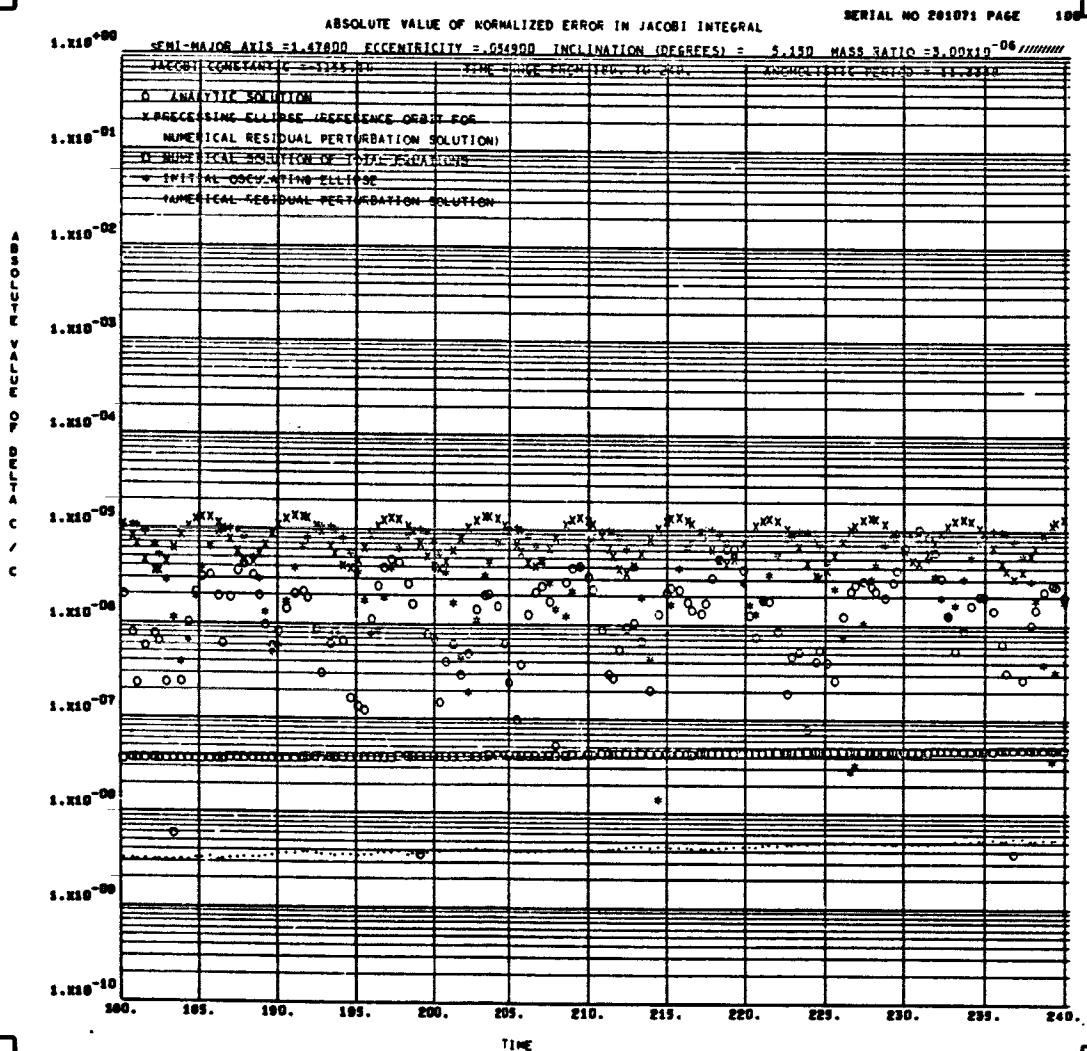


SERIAL NO 281871 PAGE 90

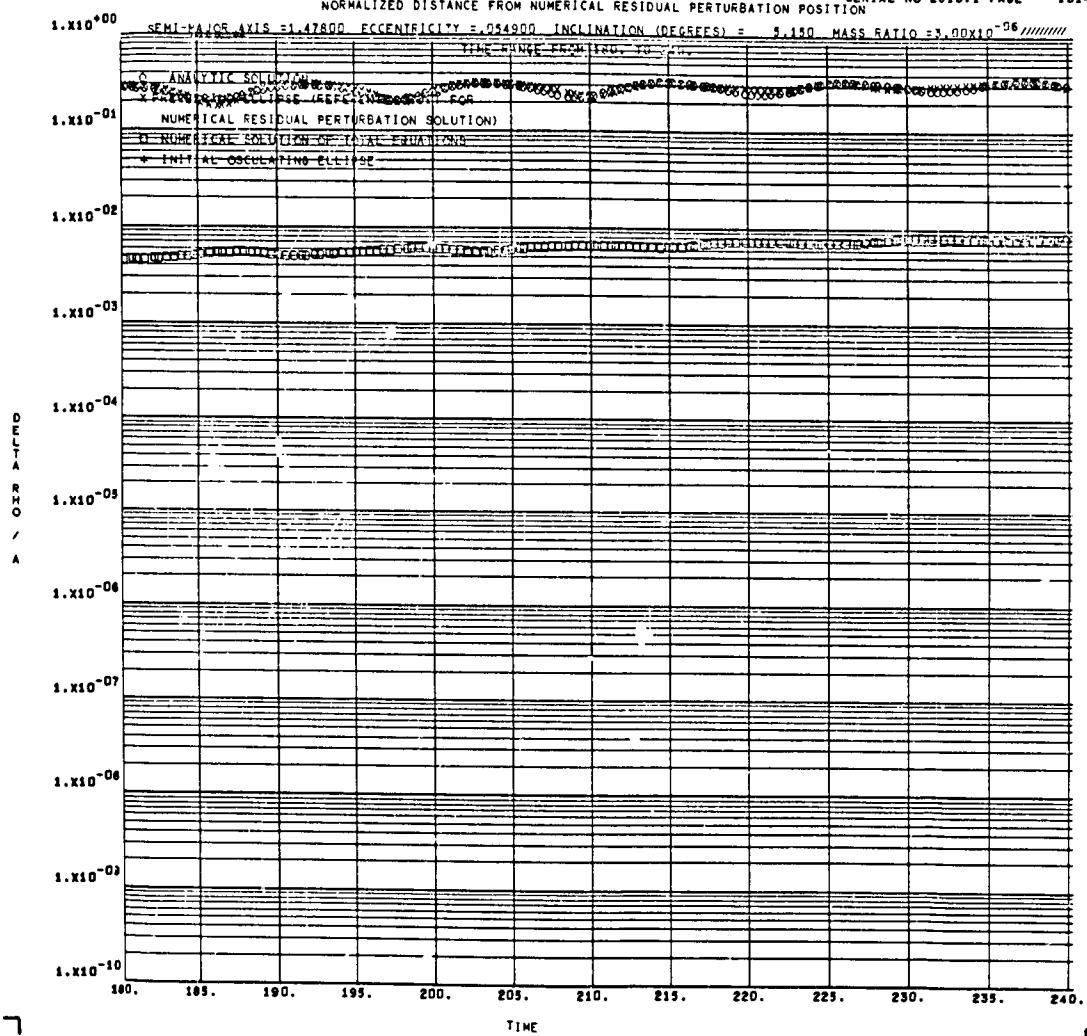


SERIAL NO 281071 PAGE 9

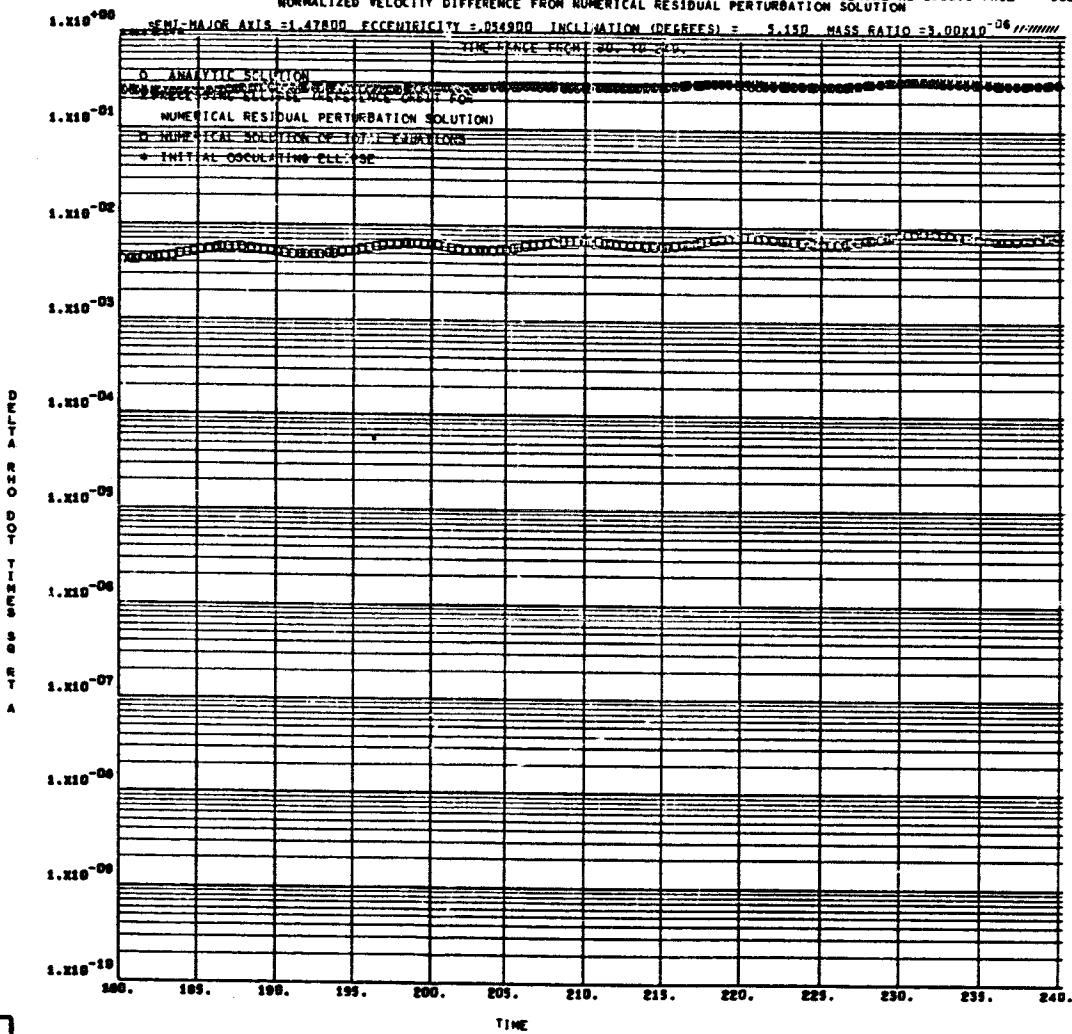




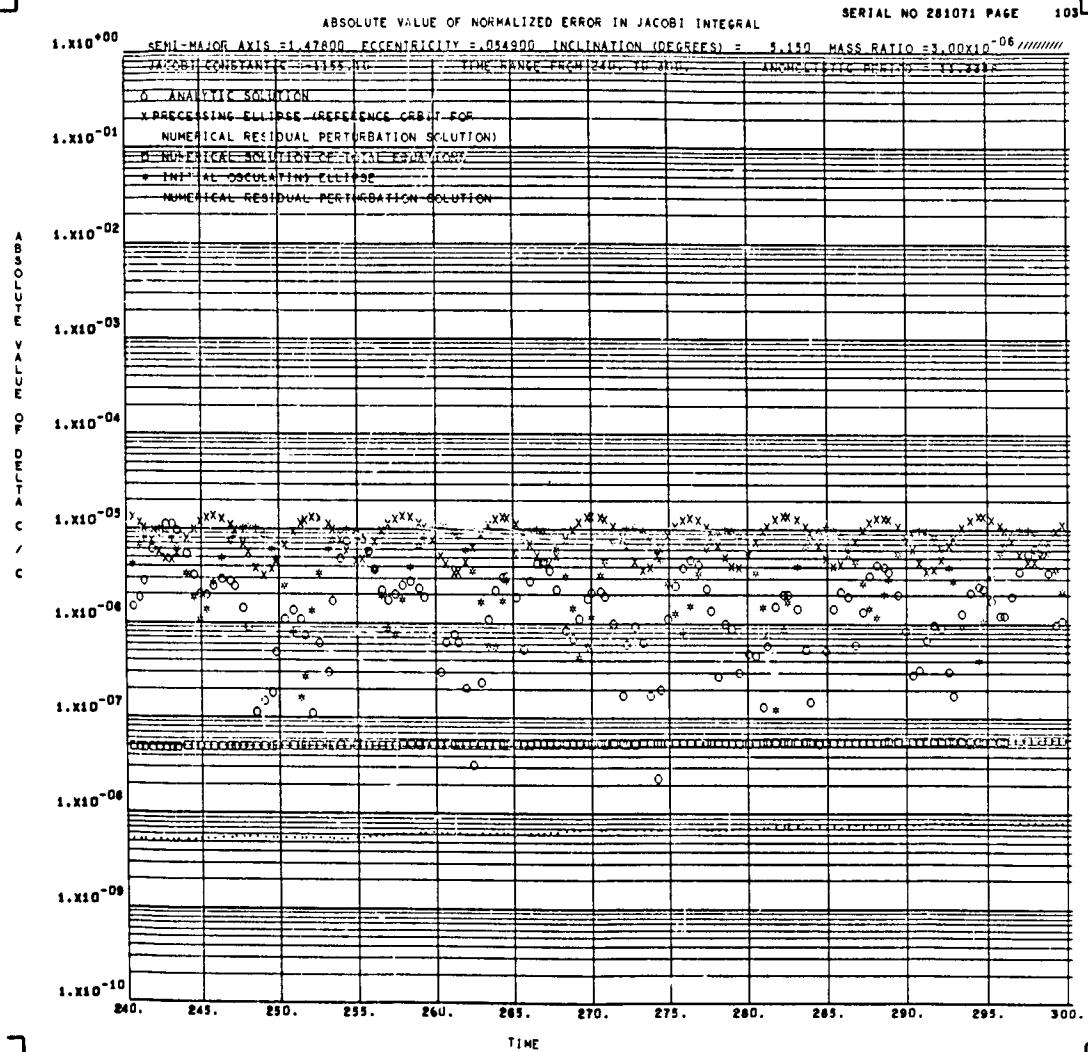
NORMALIZED DISTANCE FROM NUMERICAL RESIDUAL PERTURBATION POSITION      SERIAL NO 281071 PAGE      101



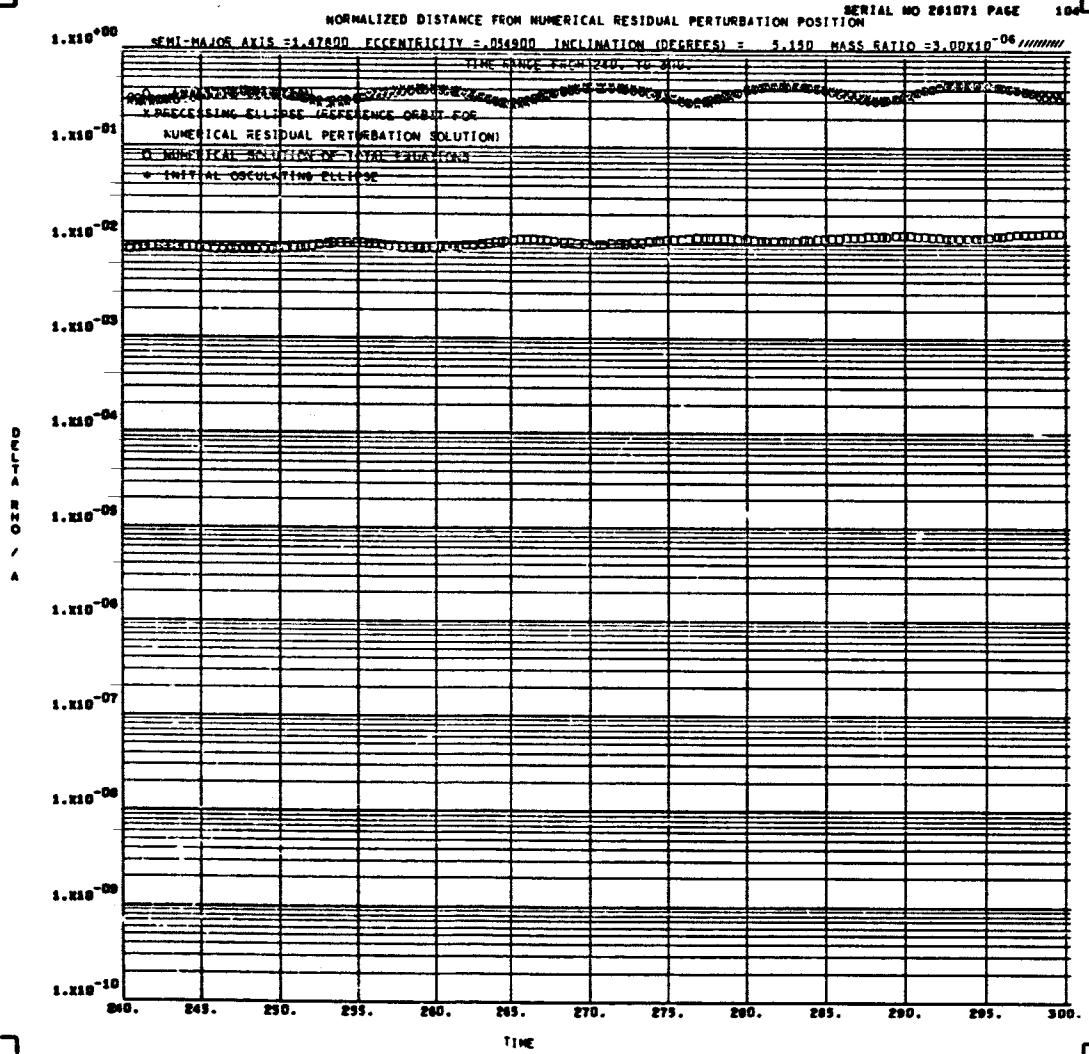
NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION



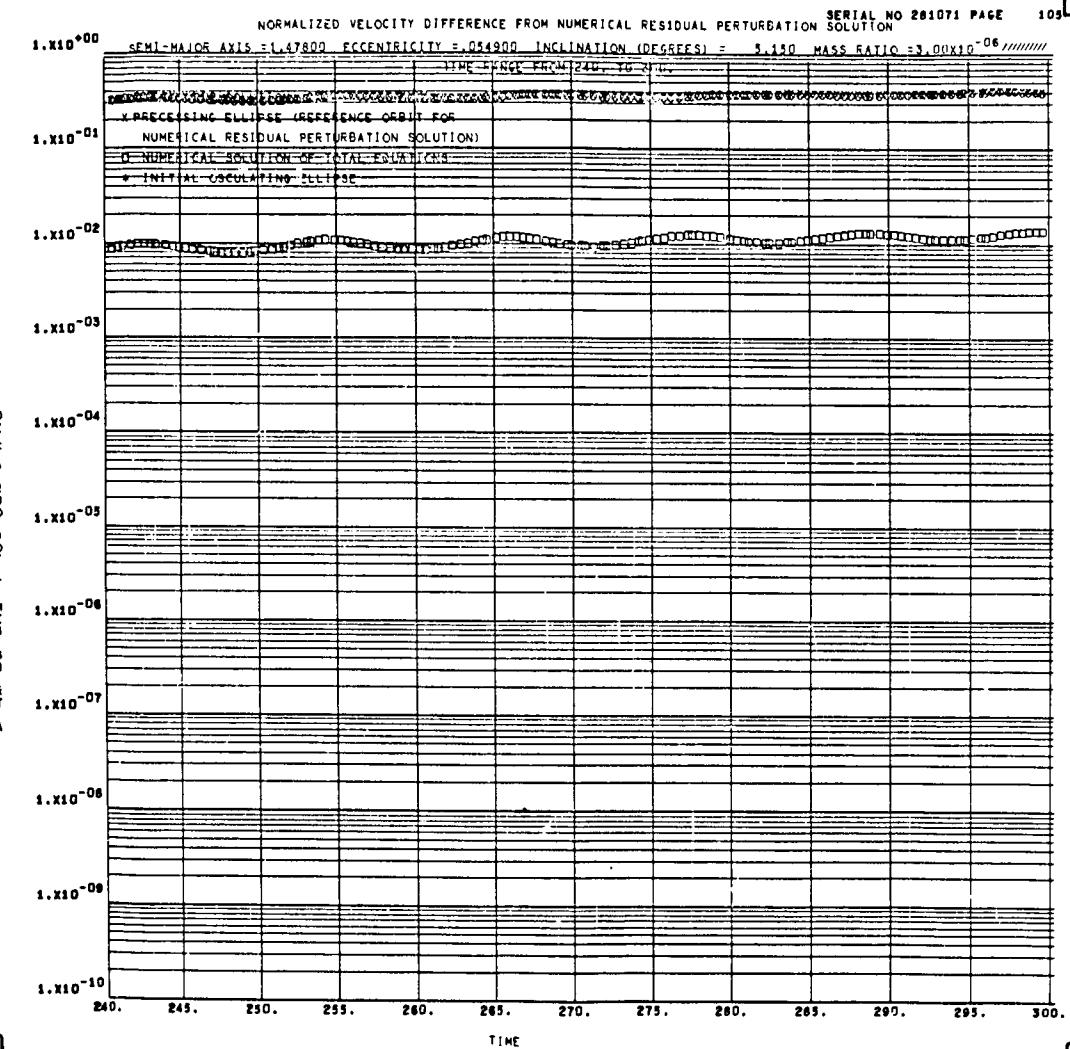
SERIAL NO 281071 PAGE 103

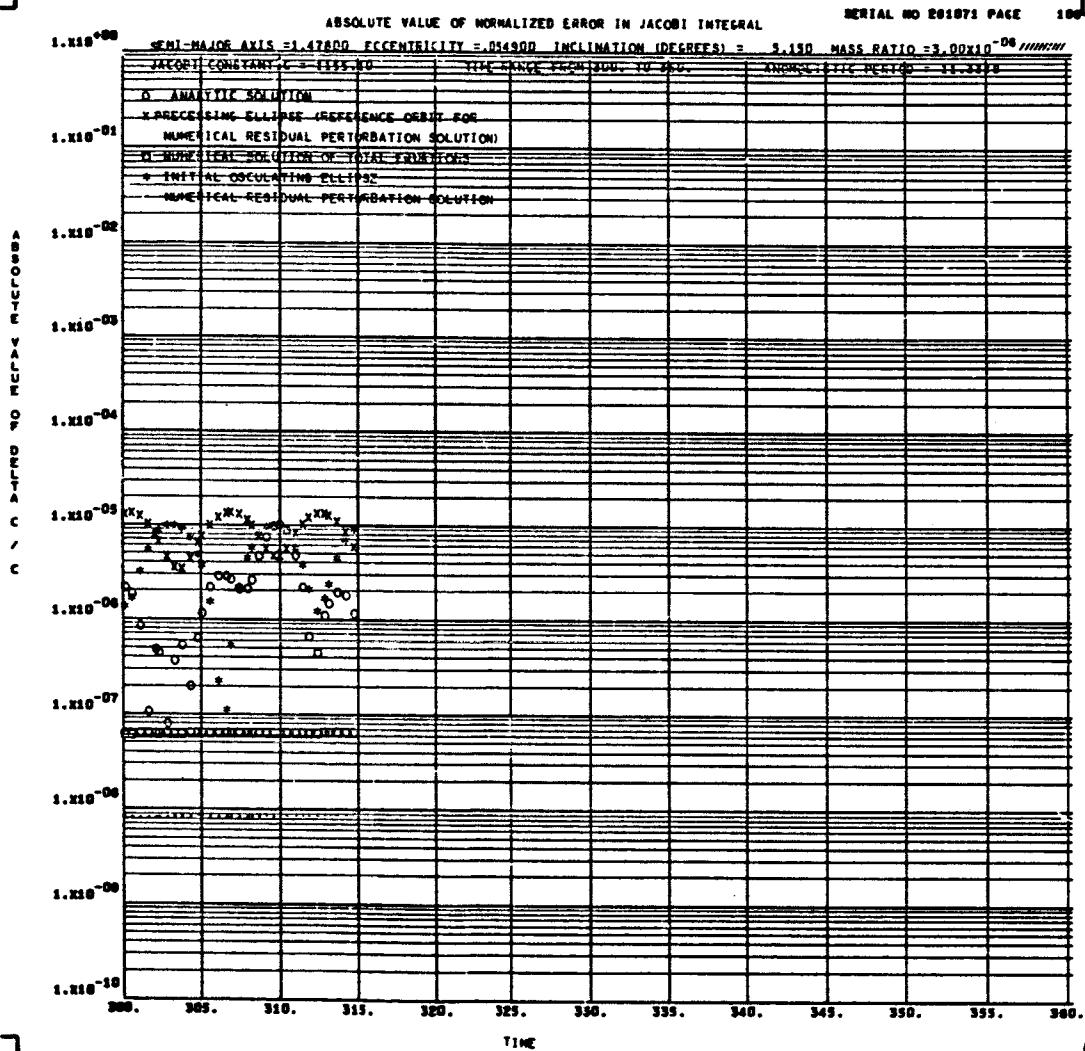


SERIAL NO 261071 PAGE 100

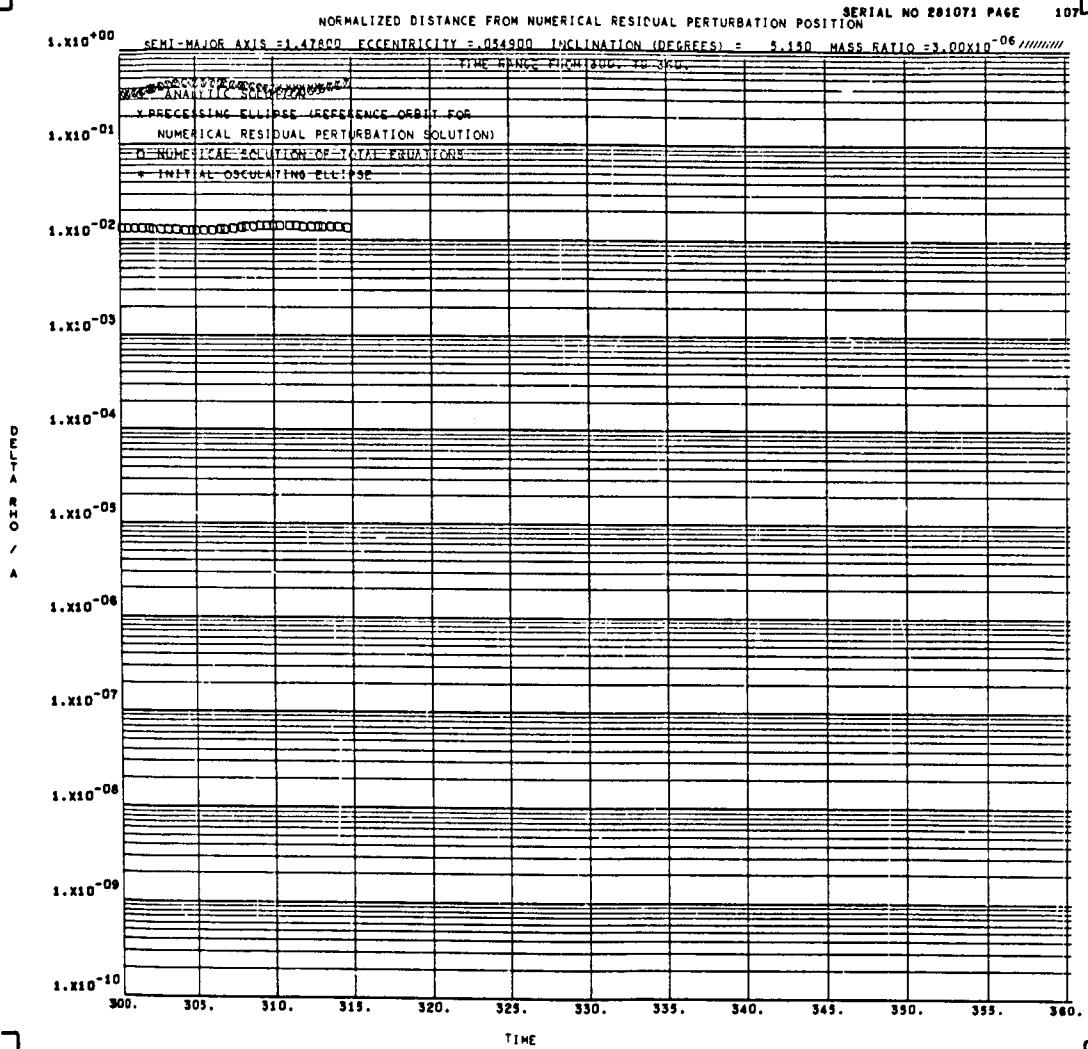


SERIAL NO 281071 PAGE 105



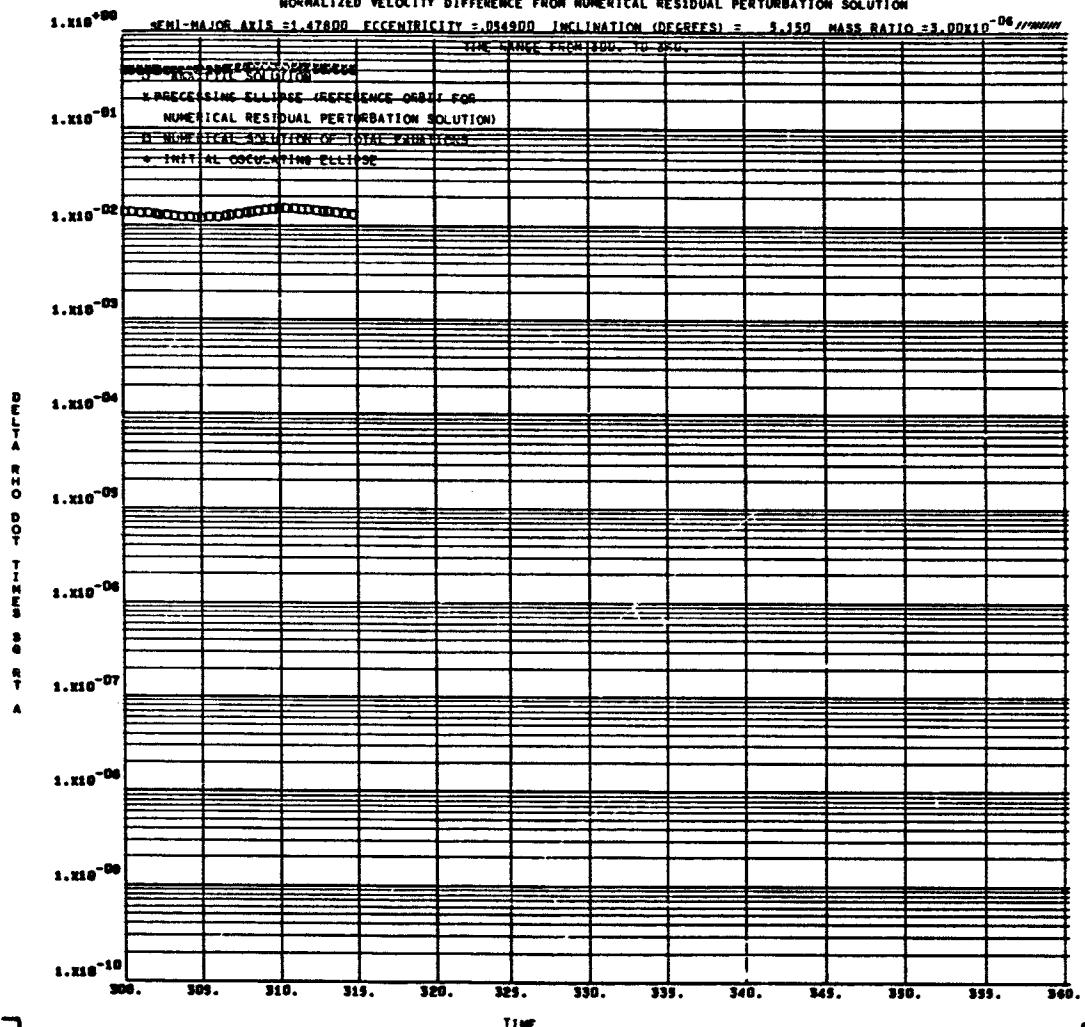


SERIAL NO 281071 PAGE 107



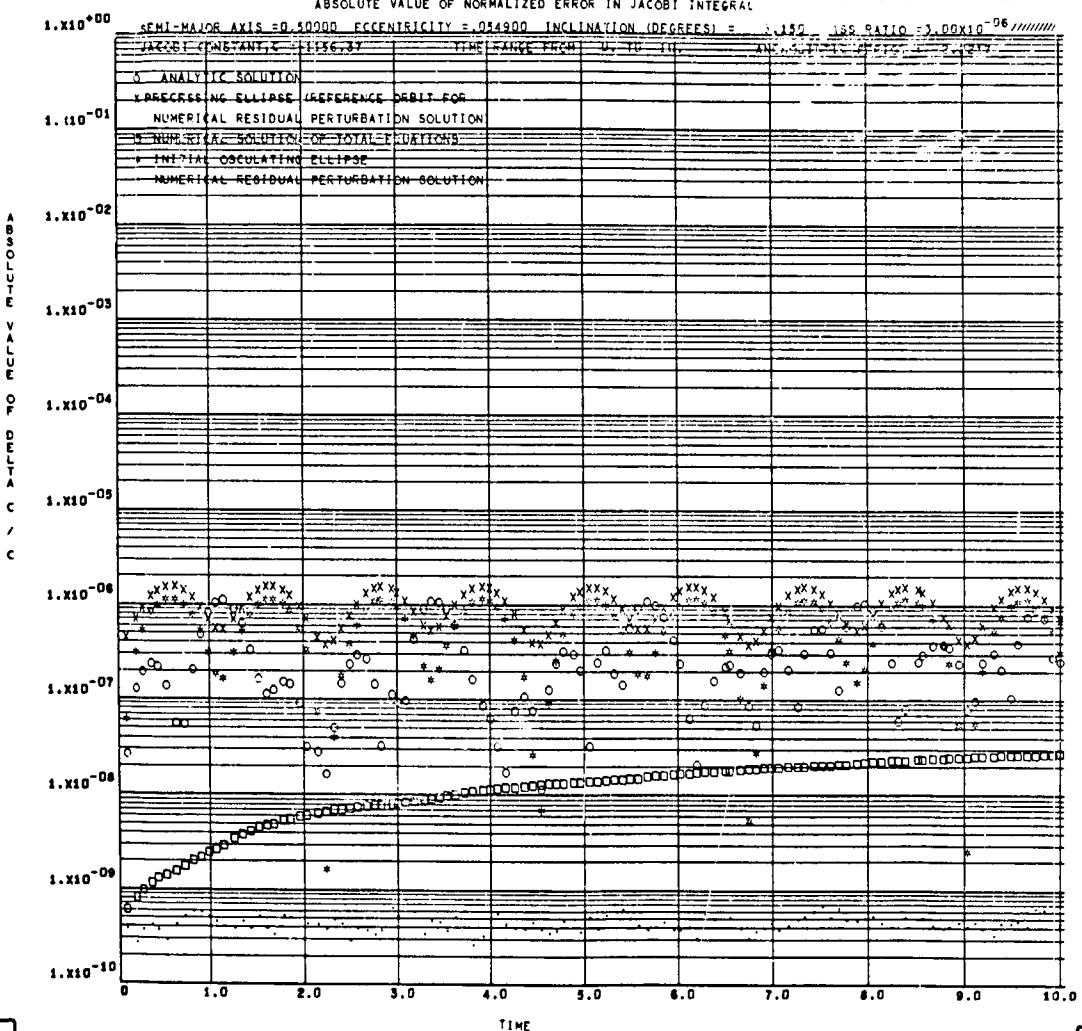
SERIAL NO 281071 PAGE 100

NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION

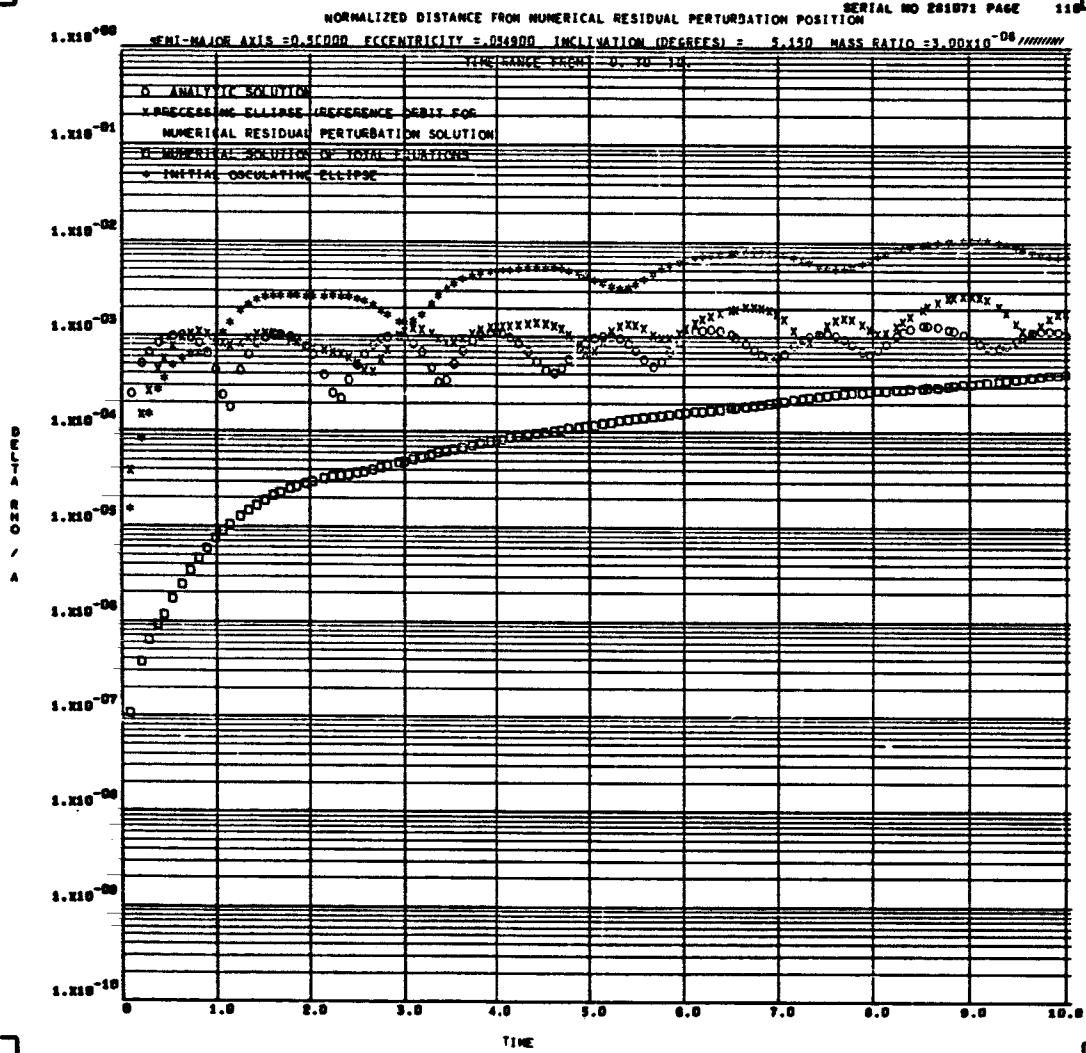


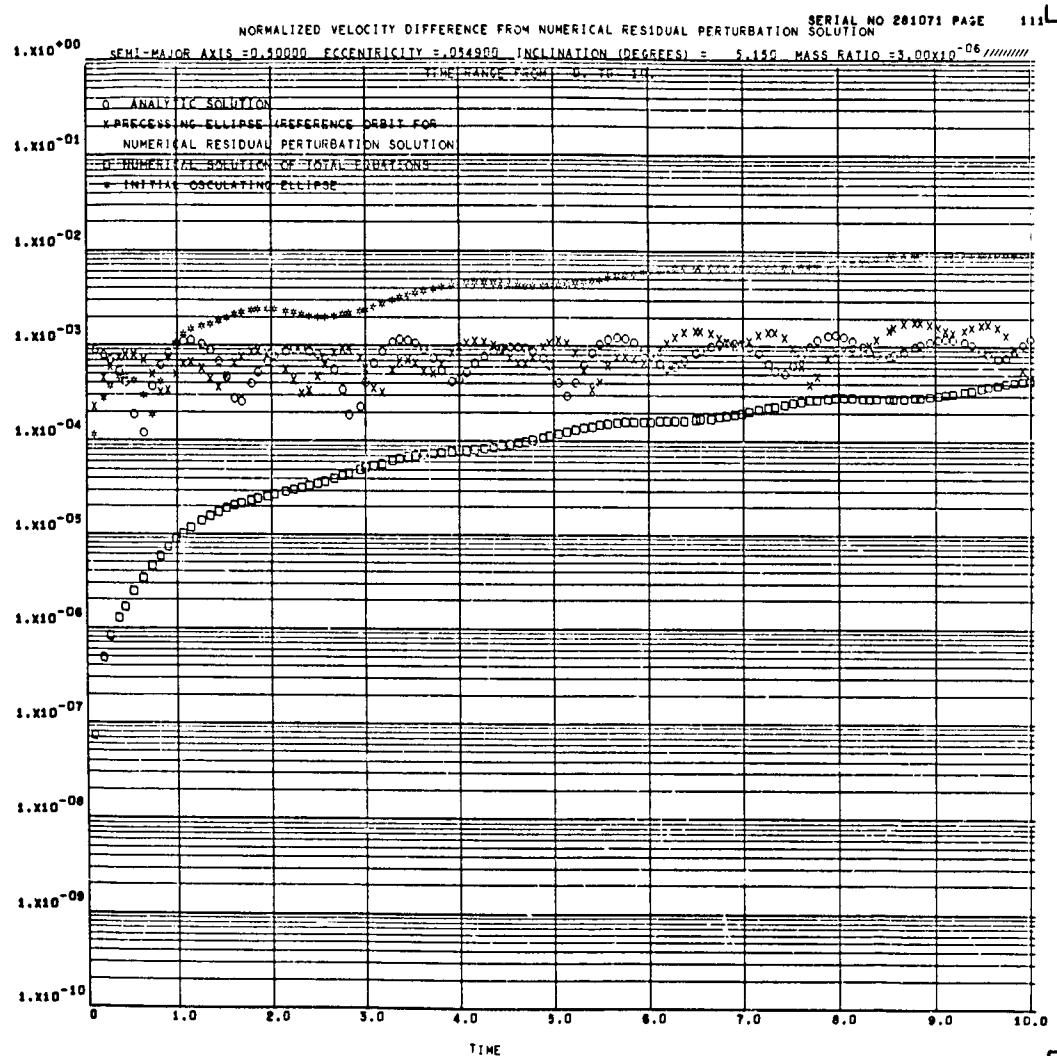
## ABSOLUTE VALUE OF NORMALIZED ERROR IN JACOBI INTEGRAL

SERIAL NO 281071 PAGE 109

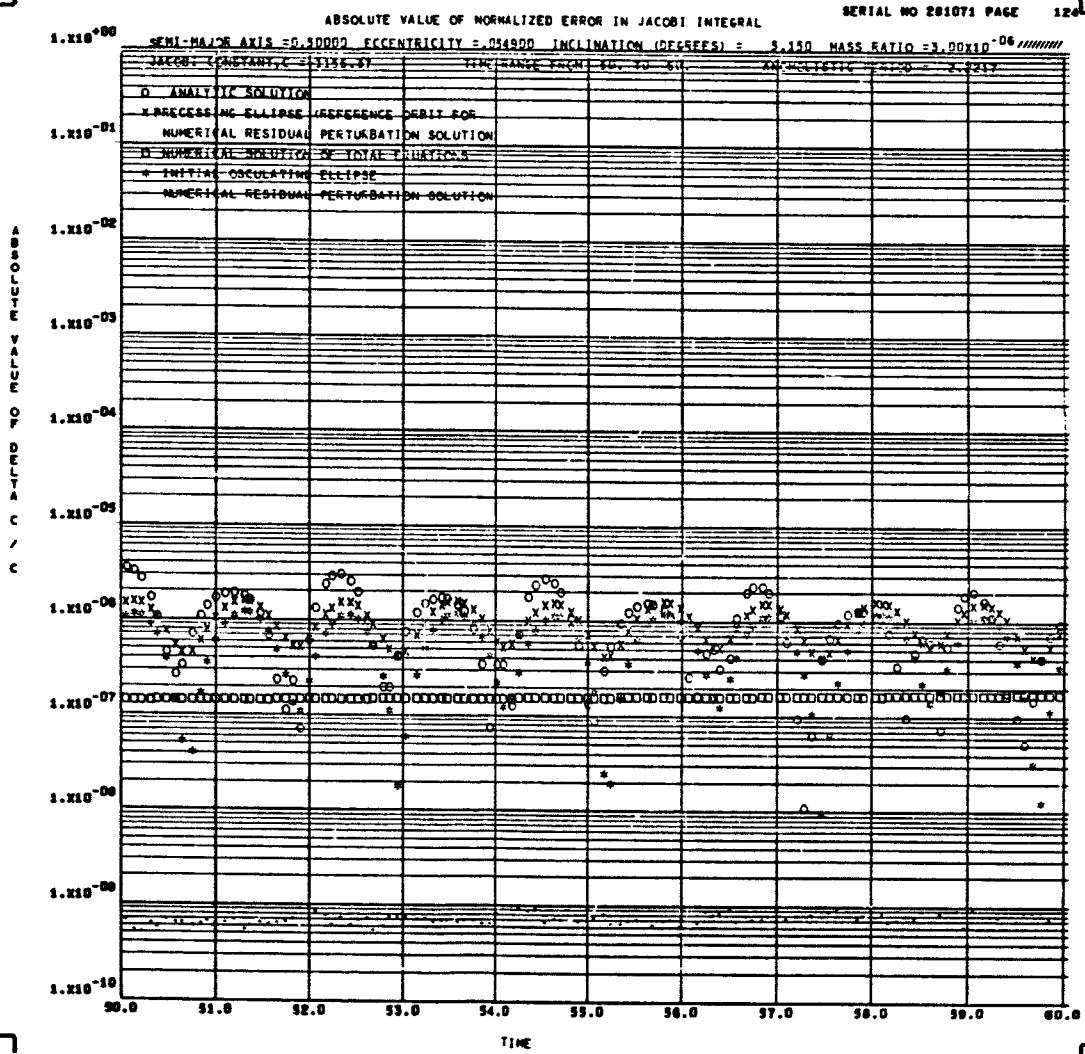


SERIAL NO 281871 PAGE 110

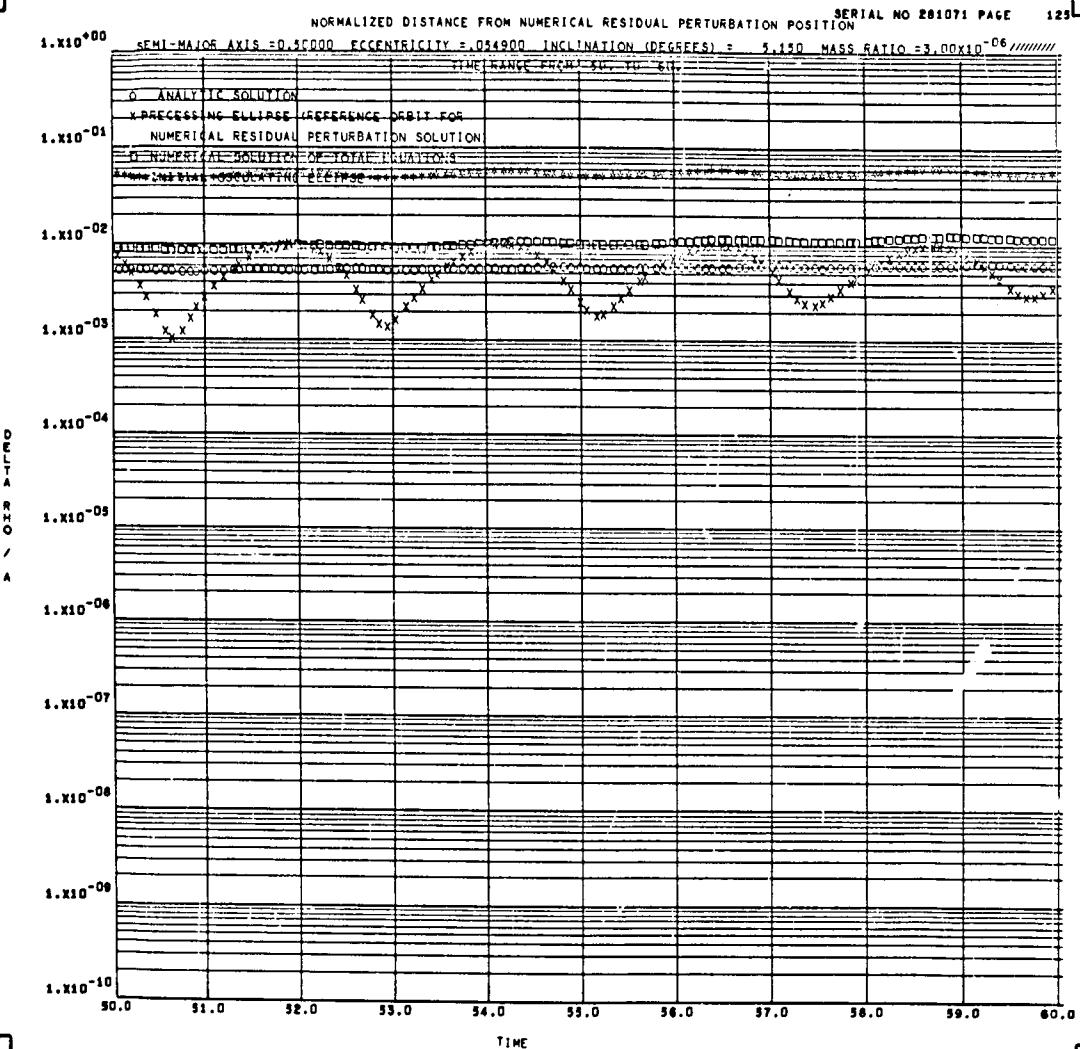




SERIAL NO 281071 PAGE 120

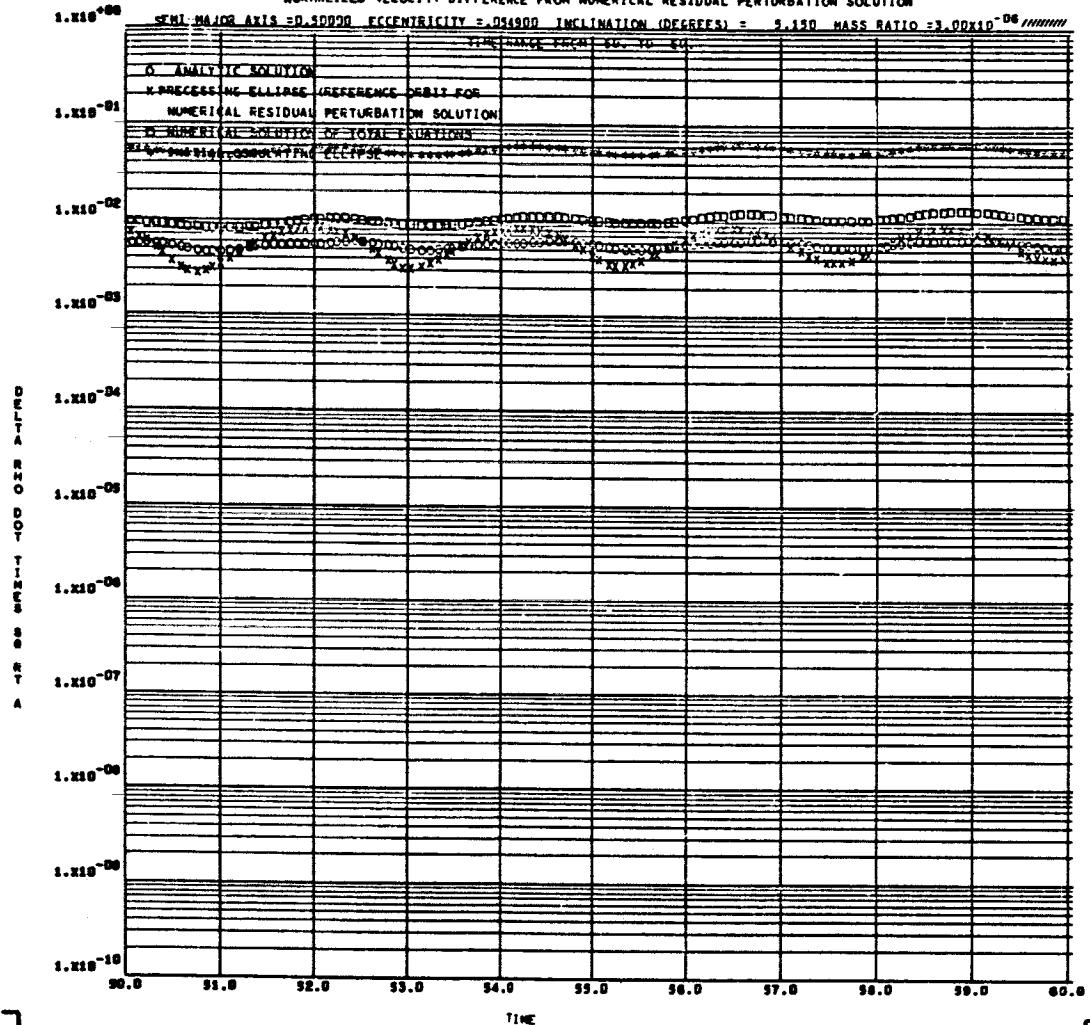


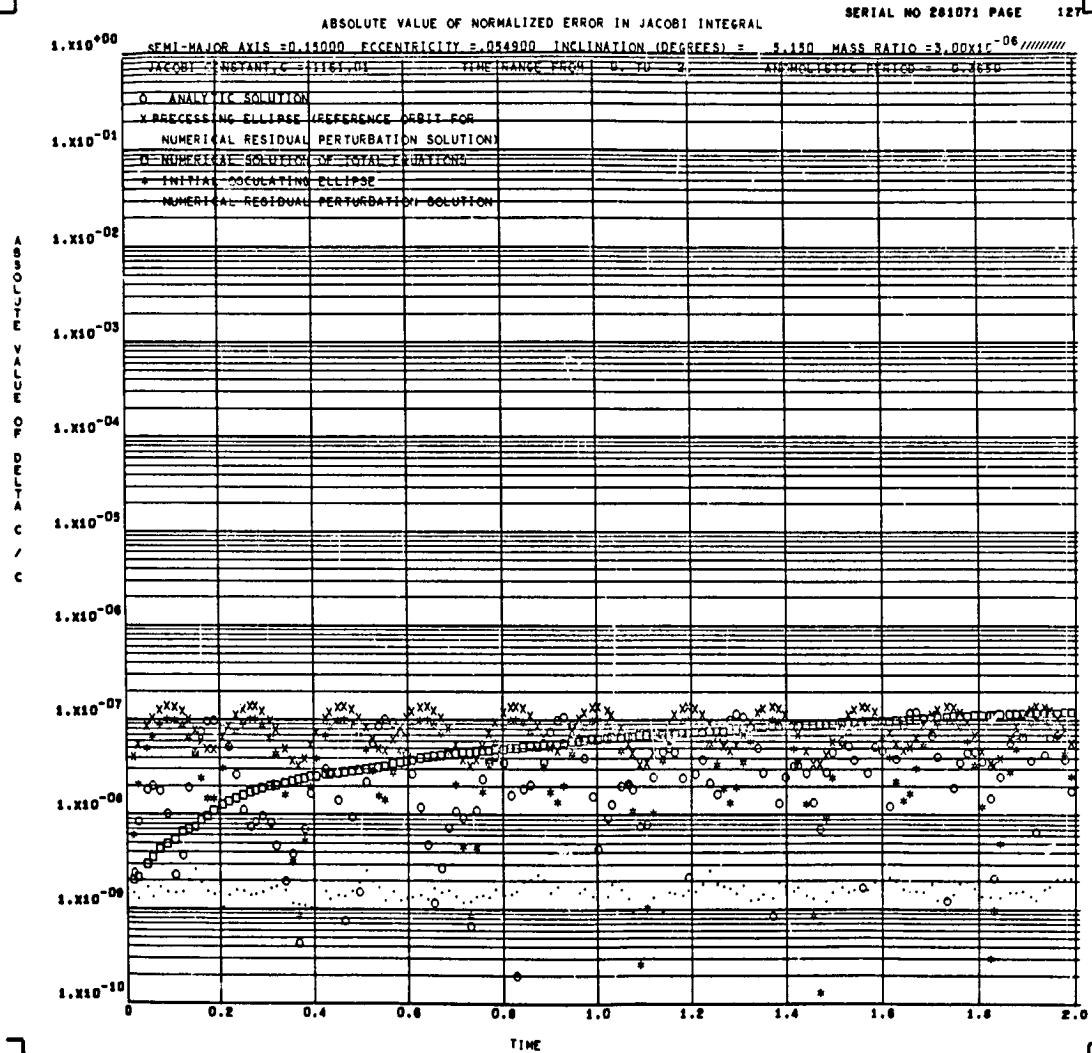
SERIAL NO 281071 PAGE 125

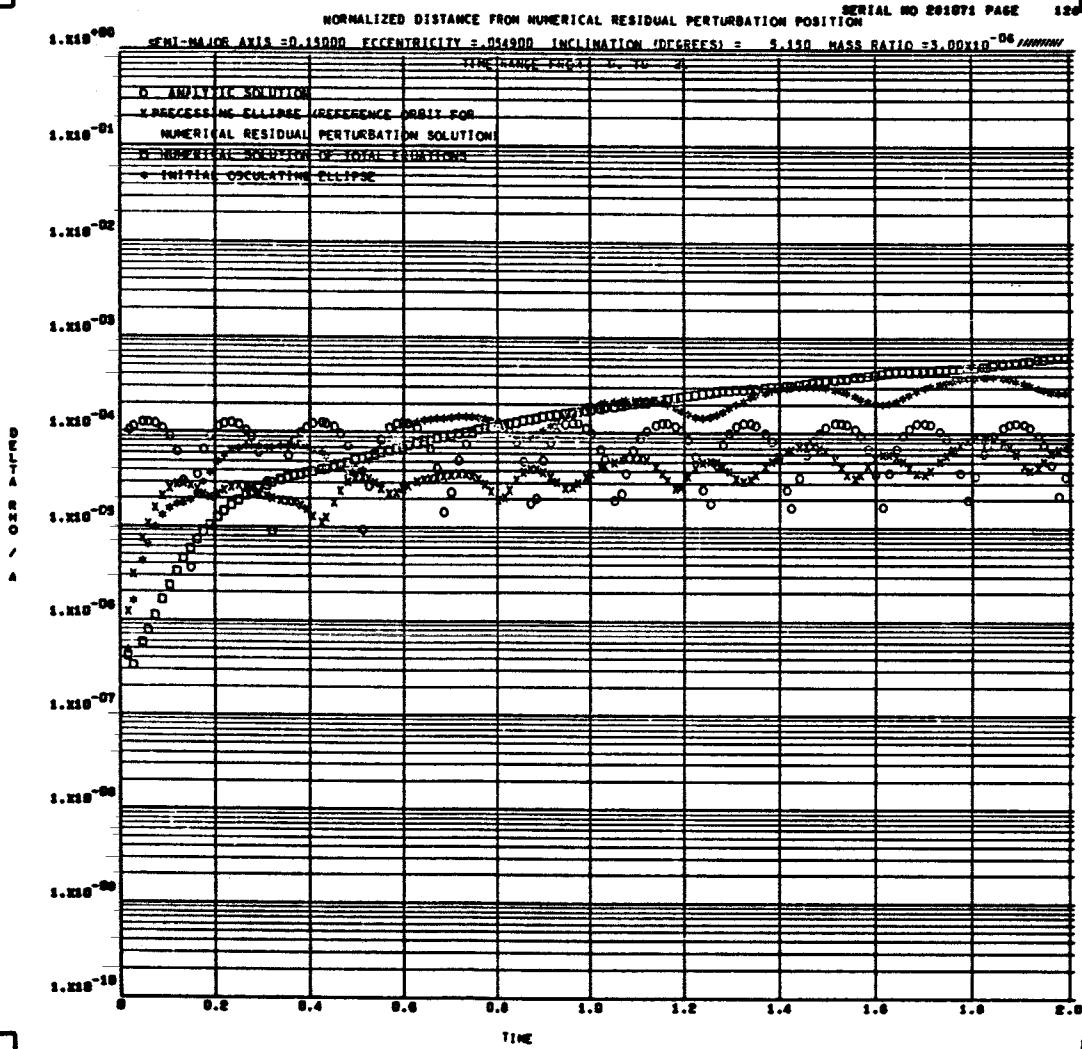


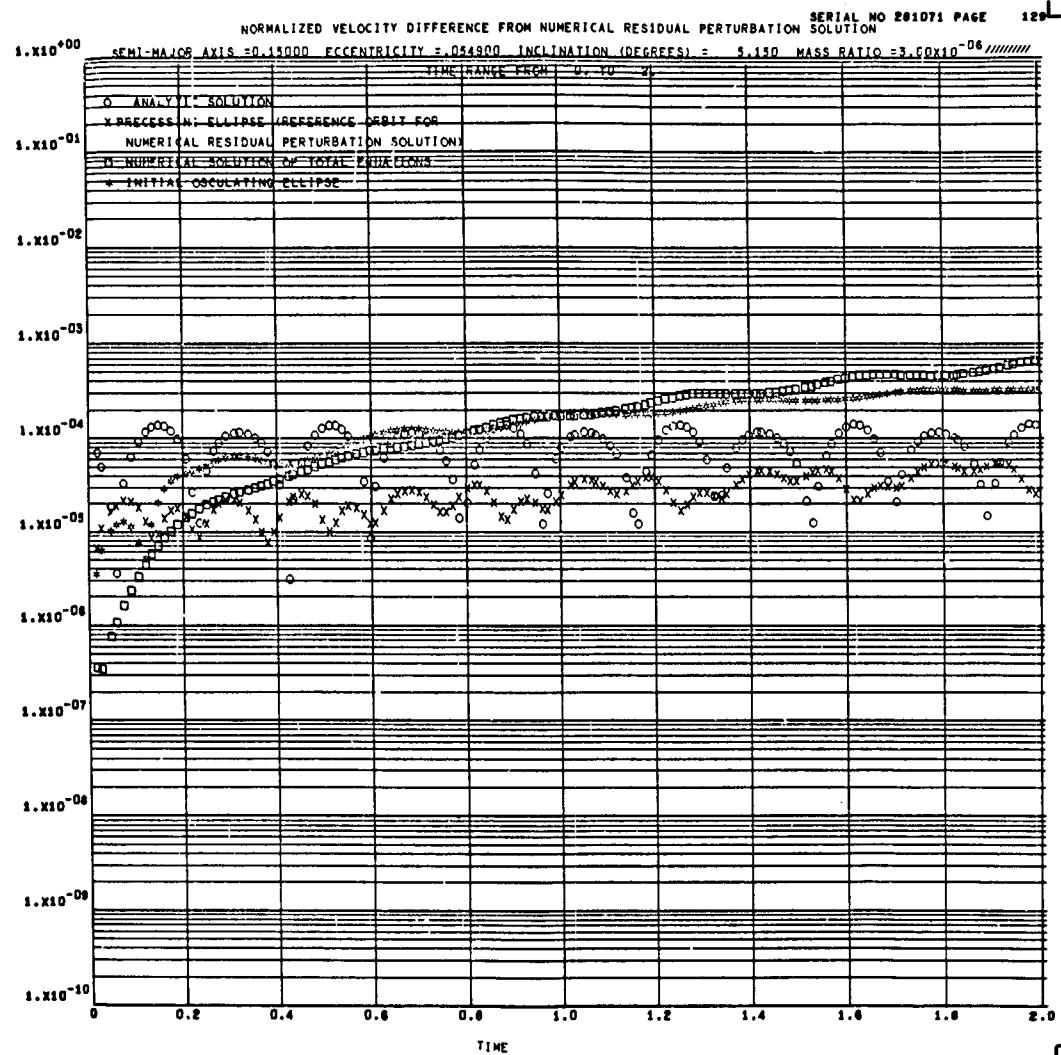
NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION

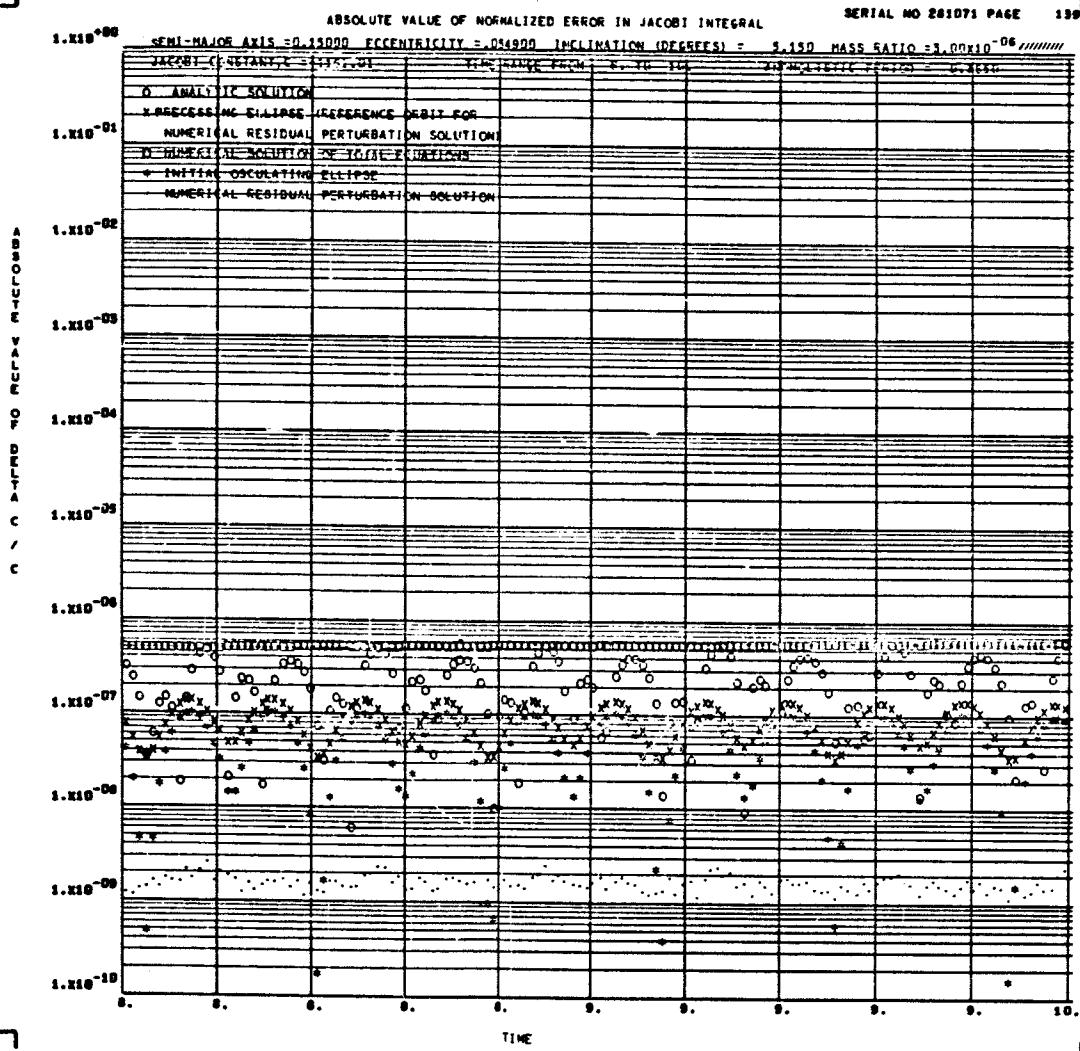
120





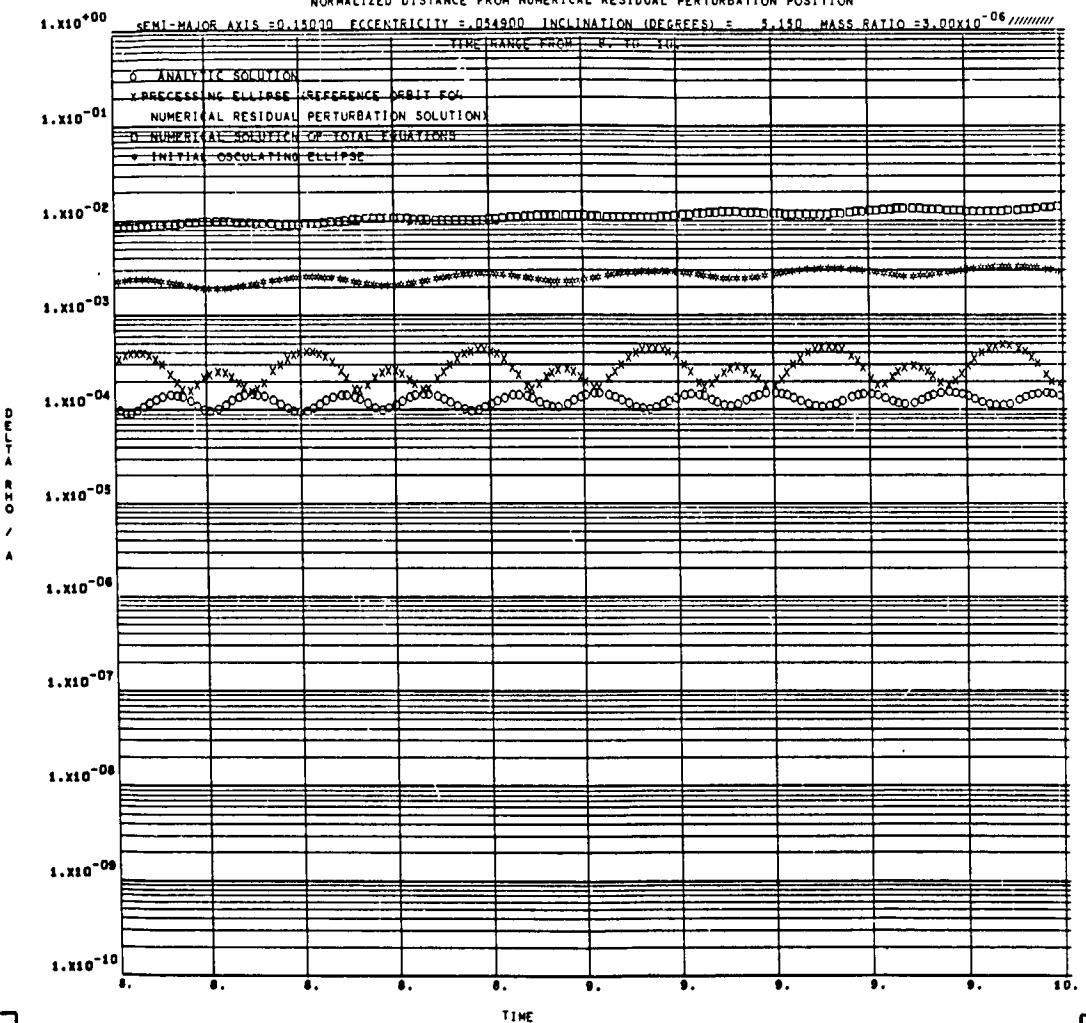






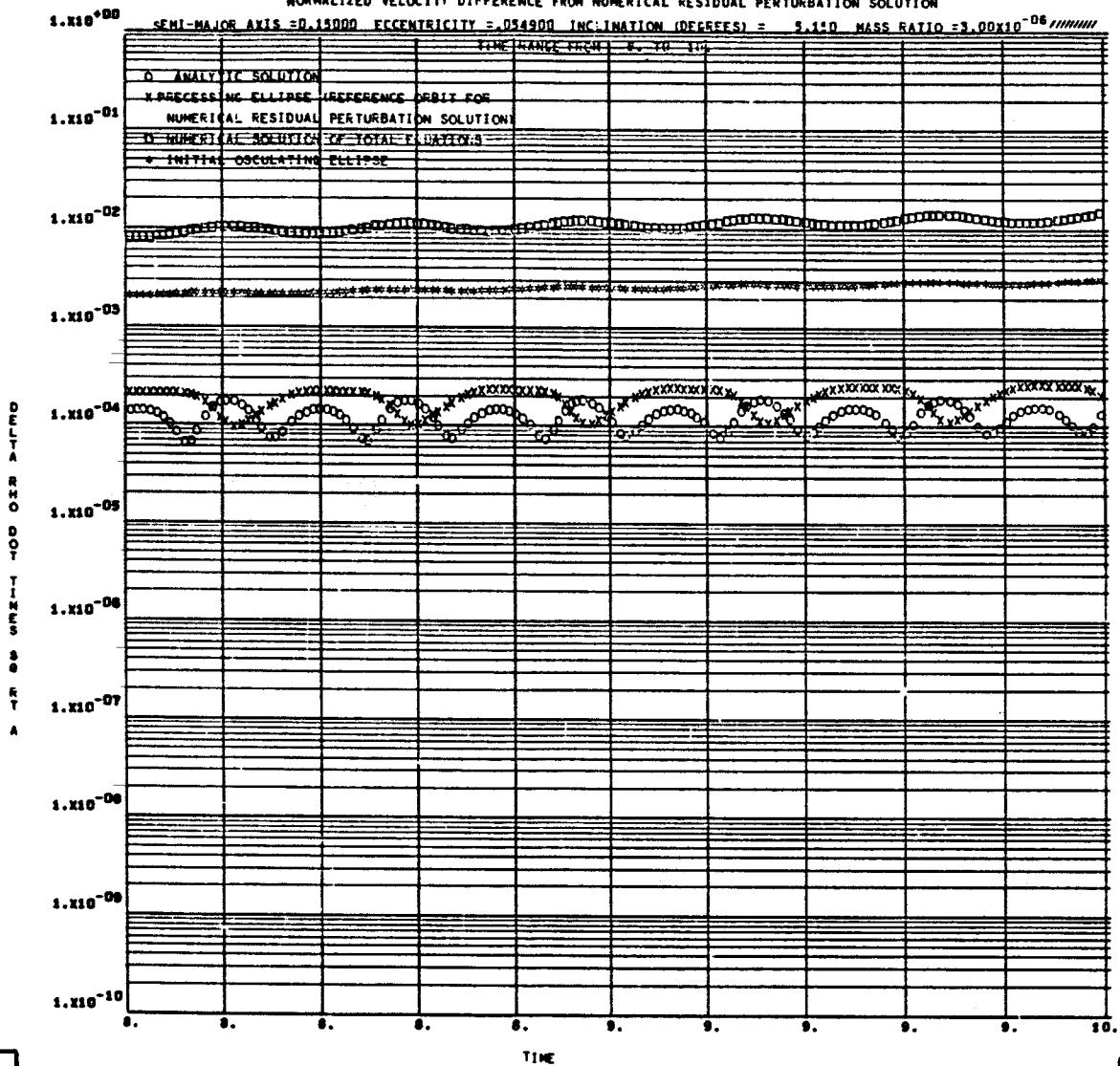
NORMALIZED DISTANCE FROM NUMERICAL RESIDUAL PERTURBATION POSITION

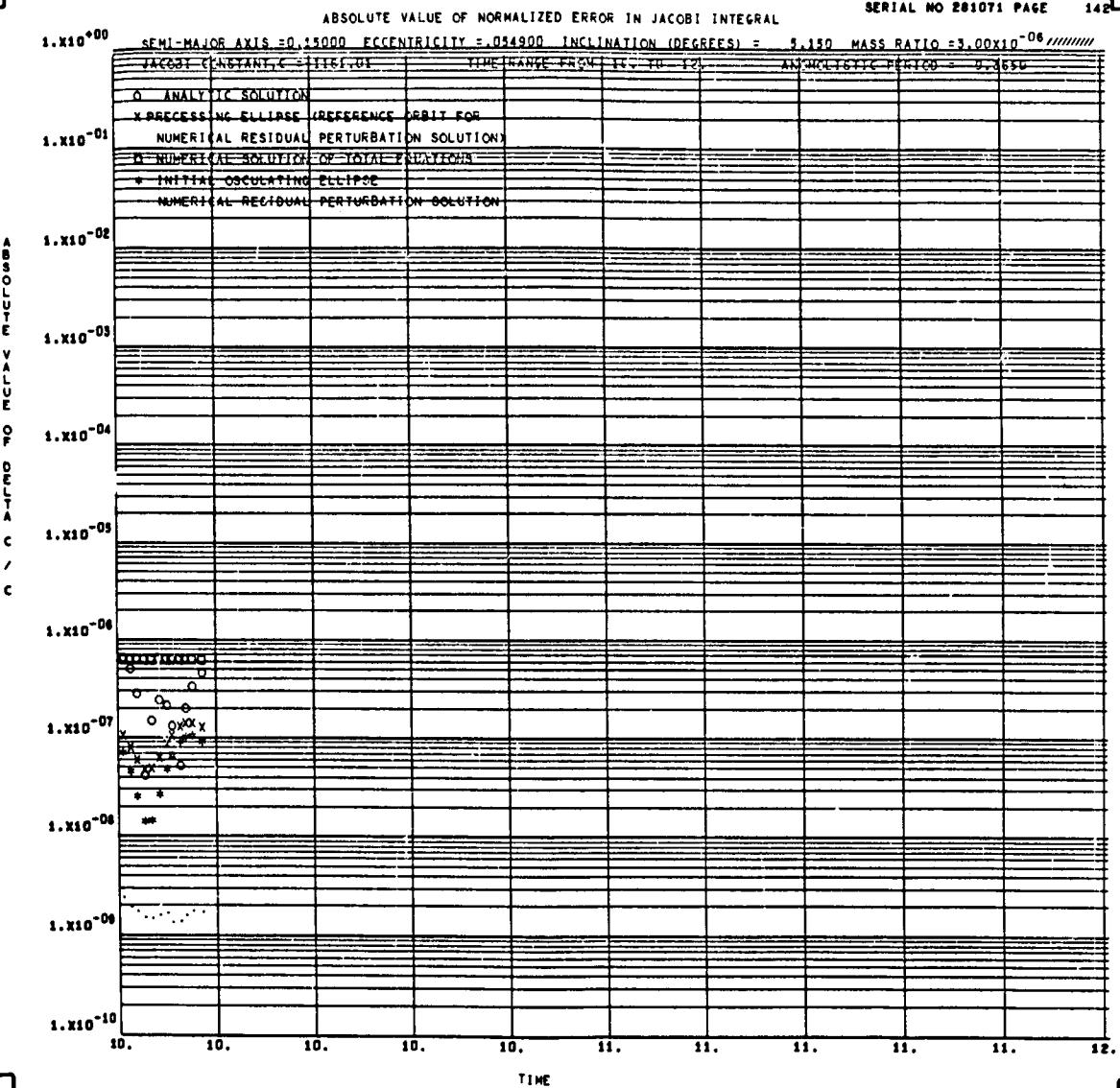
SERIAL NO 281071 PAGE 140

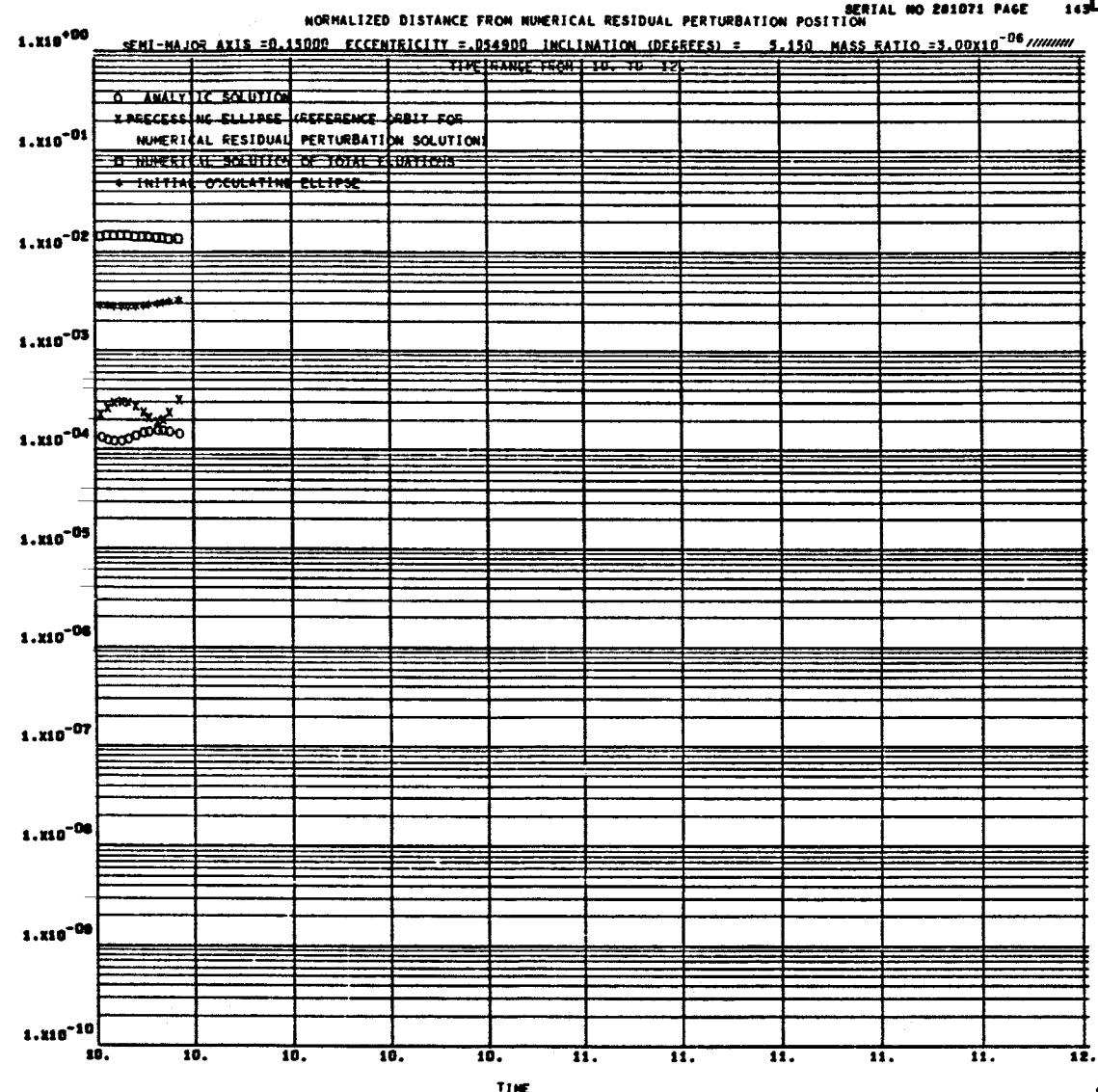


NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION

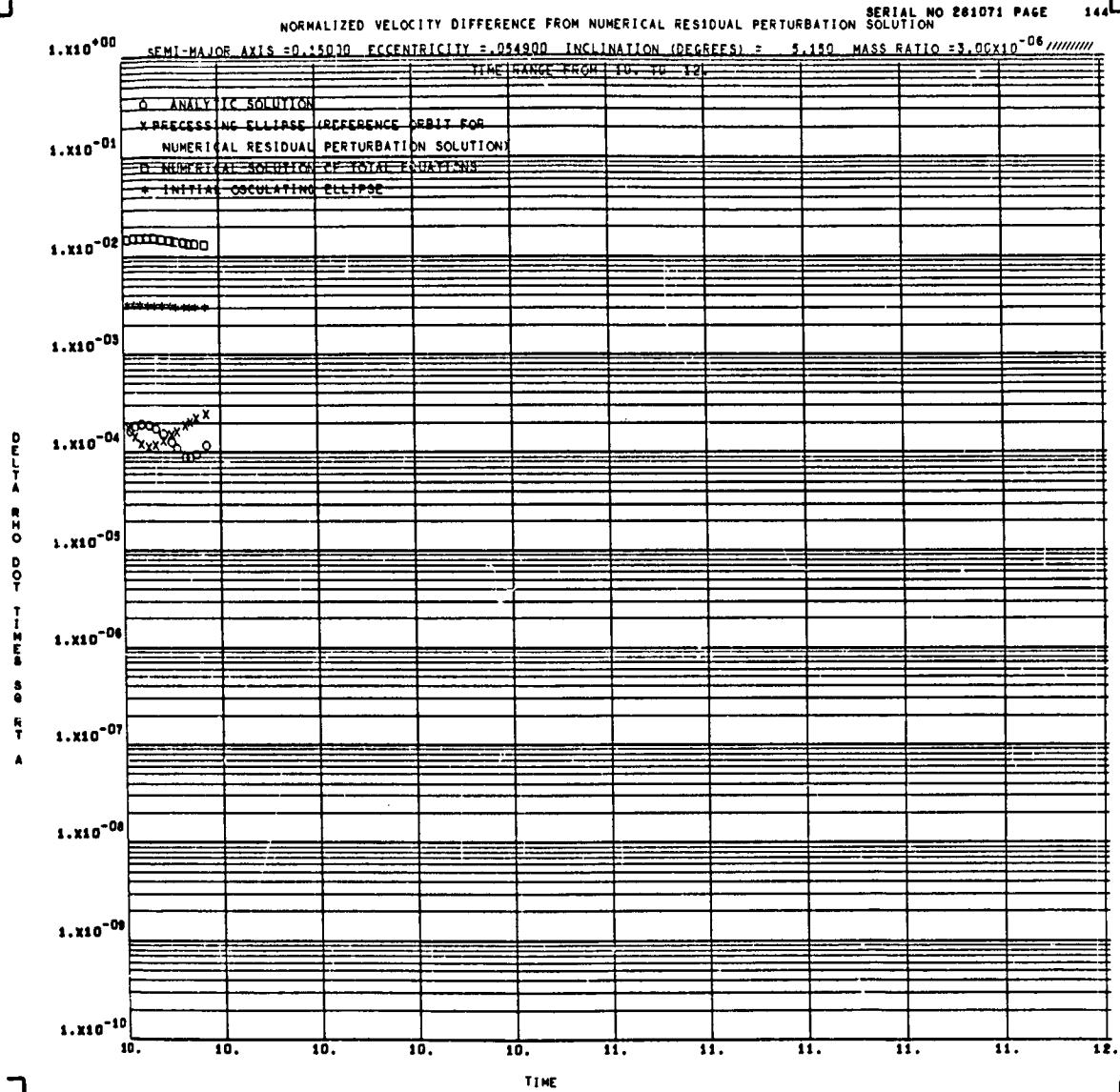
SERIAL NO 281071 PAGE 141





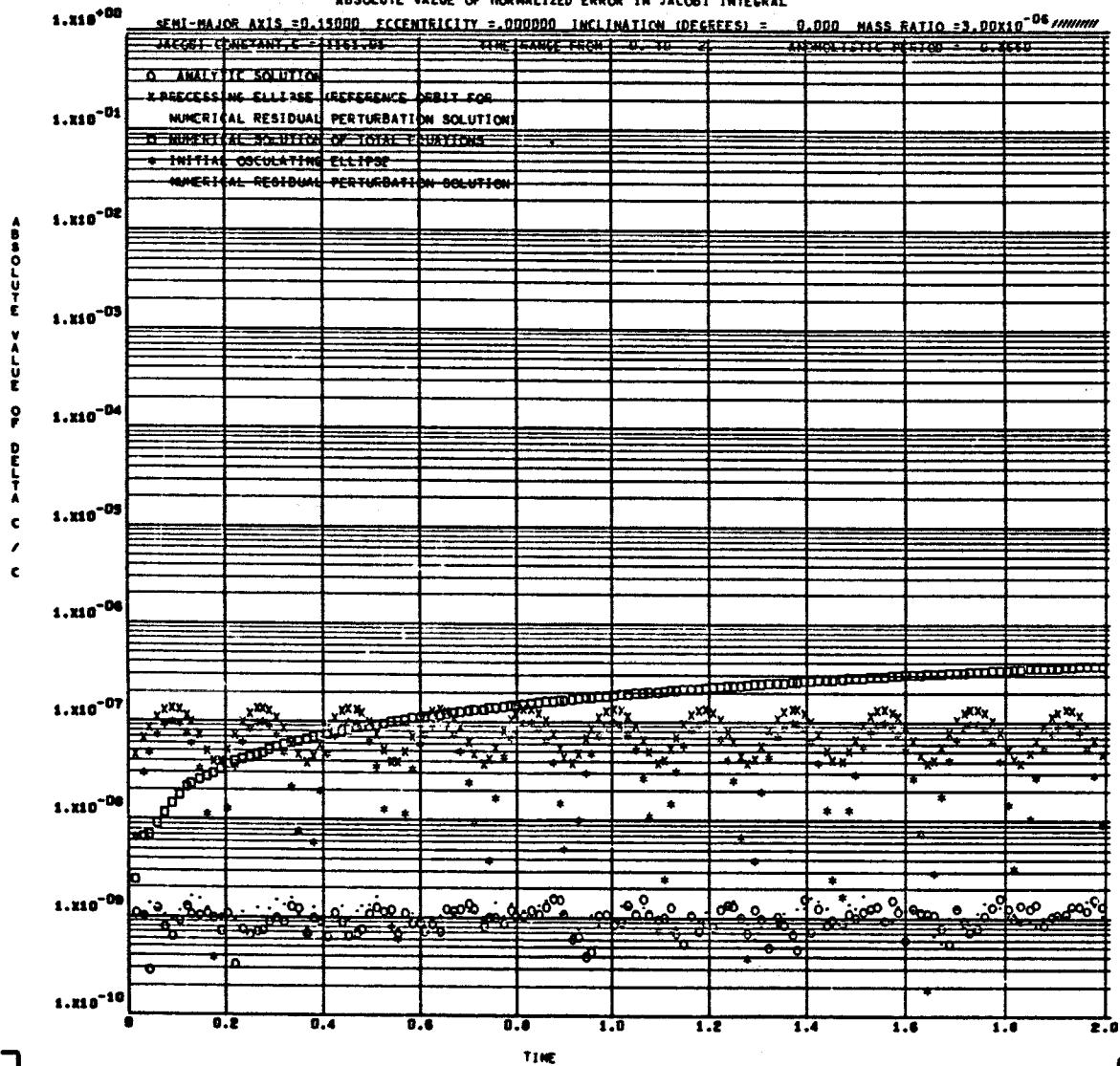


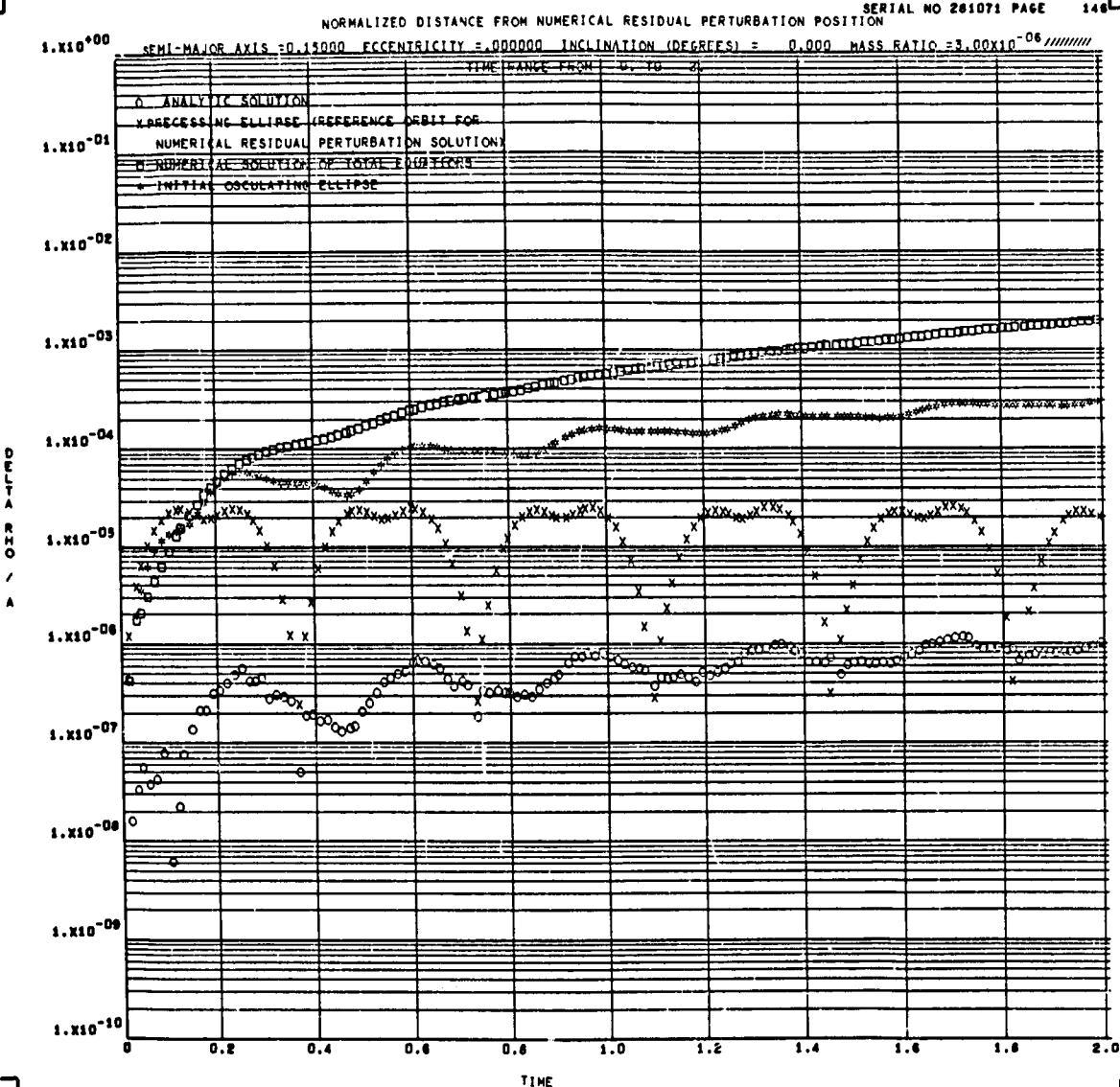
SERIAL NO 281071 PAGE 144

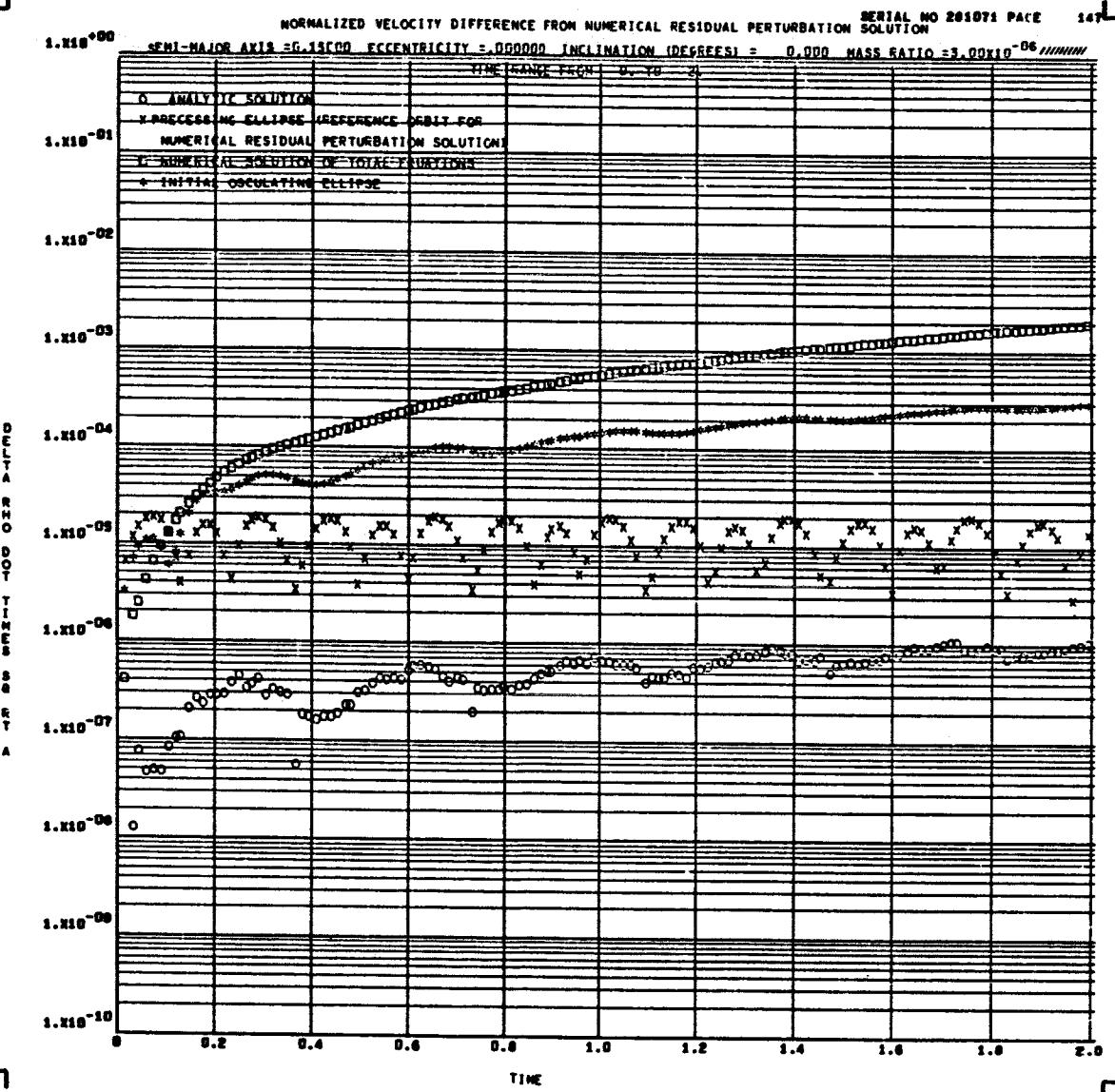


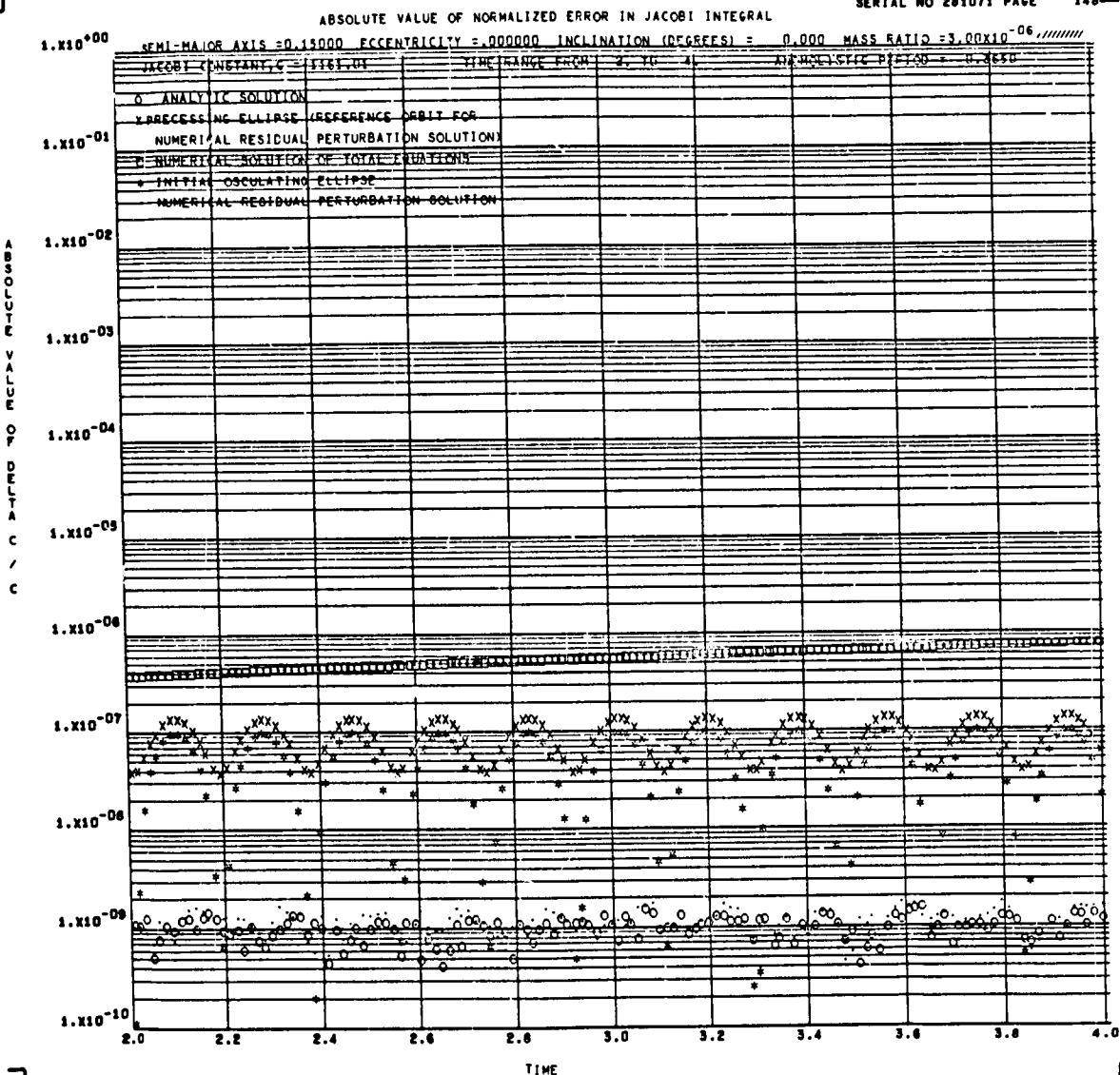
## ABSOLUTE VALUE OF NORMALIZED ERROR IN JACOBI INTEGRAL

SERIAL NO 261071 PAGE 145





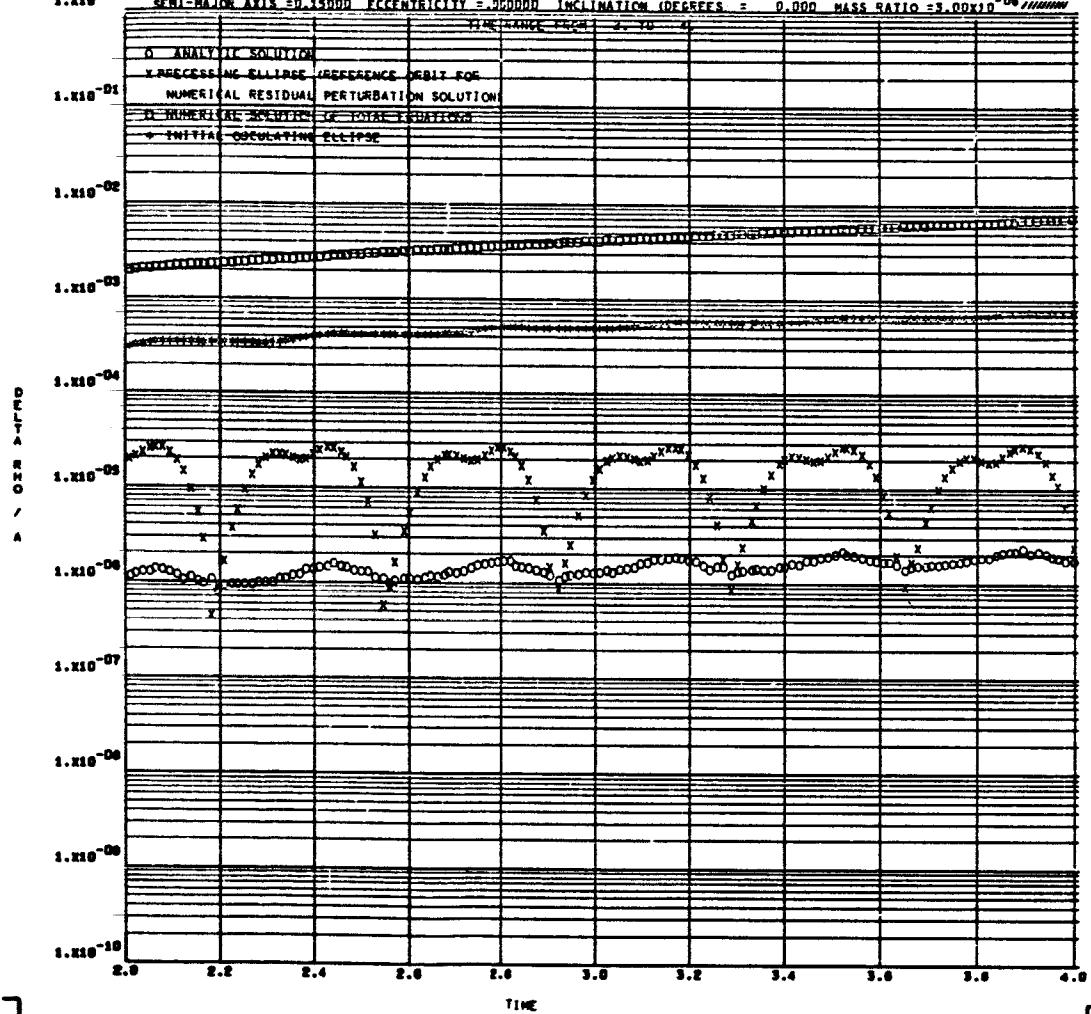




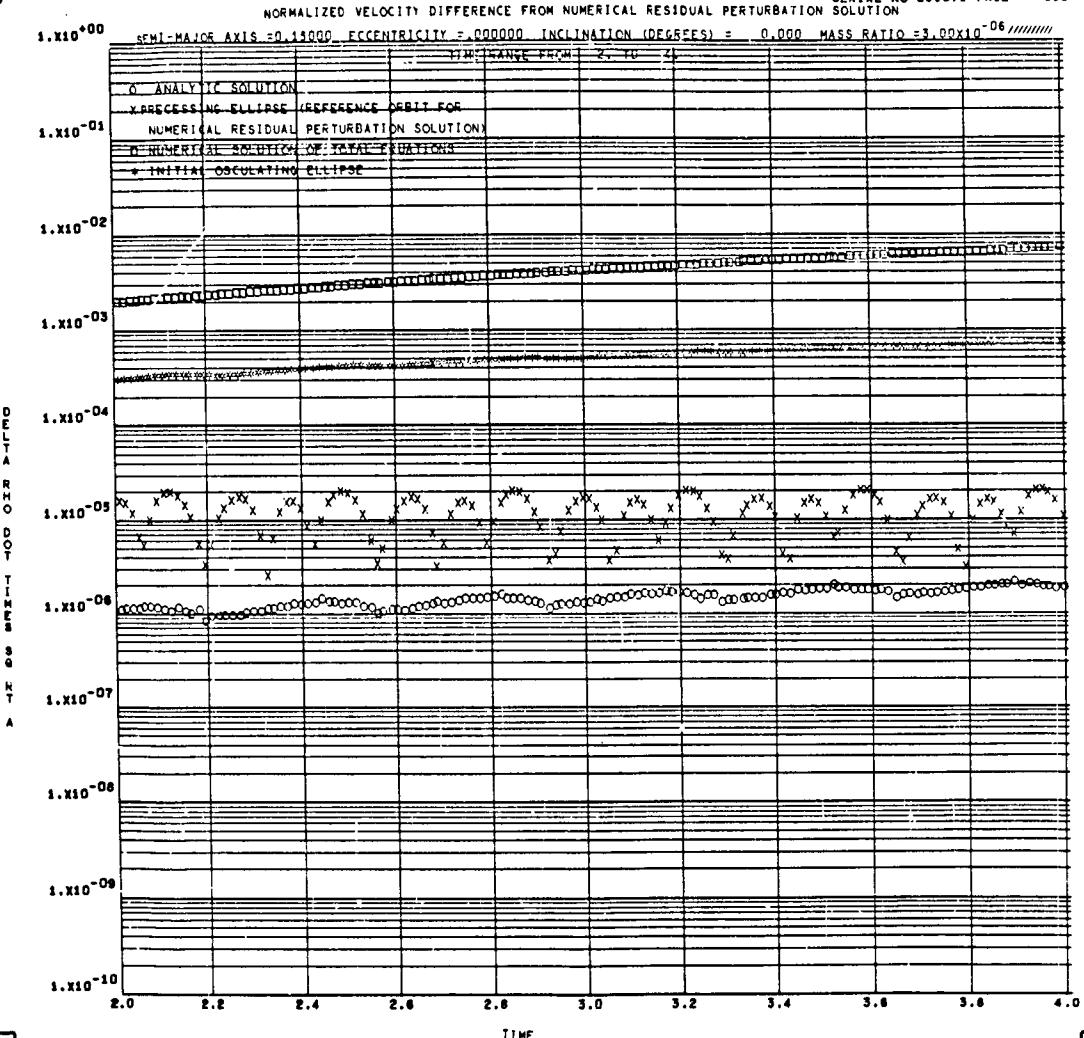
SERIAL NO 281871 PAGE 149

NORMALIZED DISTANCE FROM NUMERICAL RESIDUAL PERTURBATION POSITION

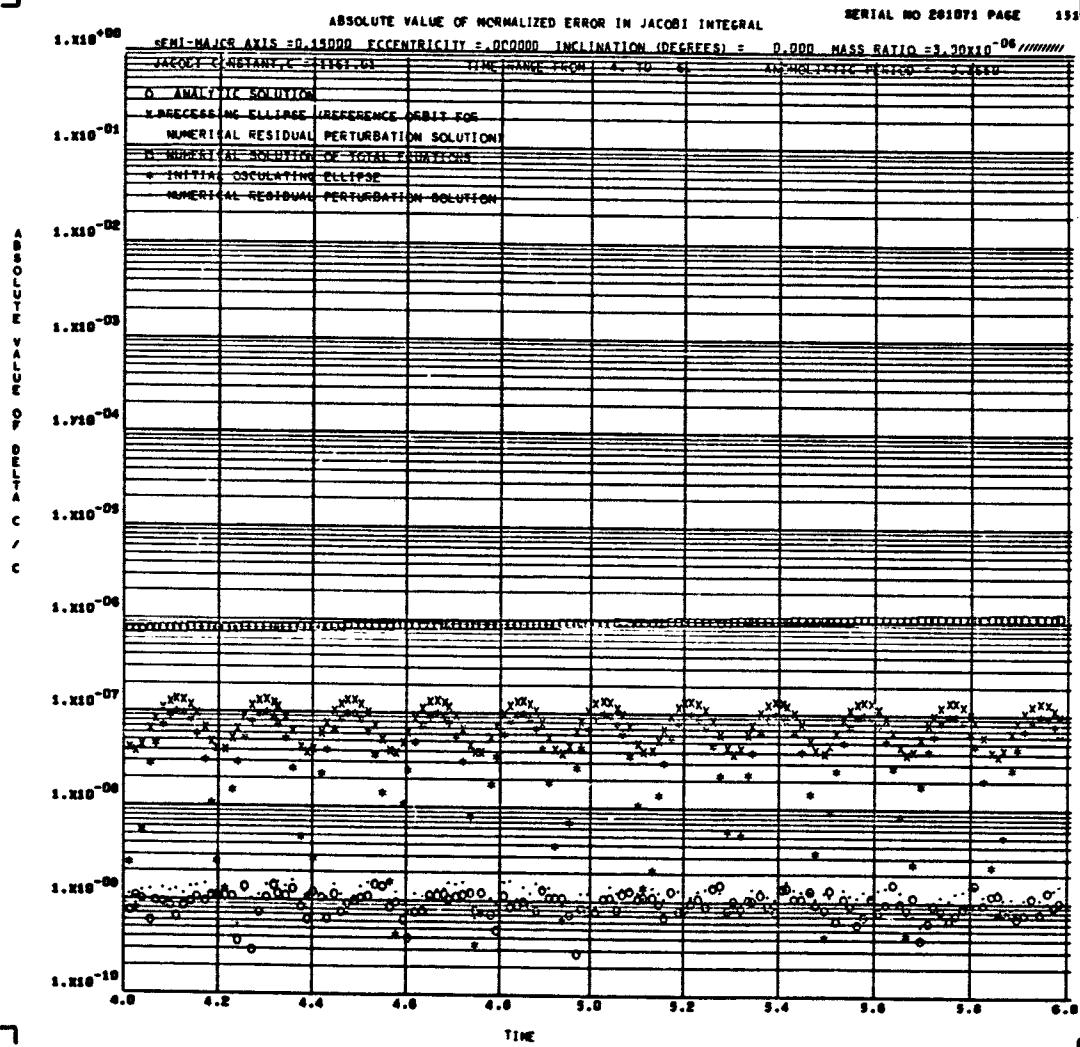
~~1.E16<sup>-06</sup> SEMI-MAJOR AXIS = 0.150000 ECCENTRICITY = .000000 INCLINATION (DEGREES) = 0.000 MASS RATIO = 3.000; D=06~~



SERIAL NO 231071 PAGE 150

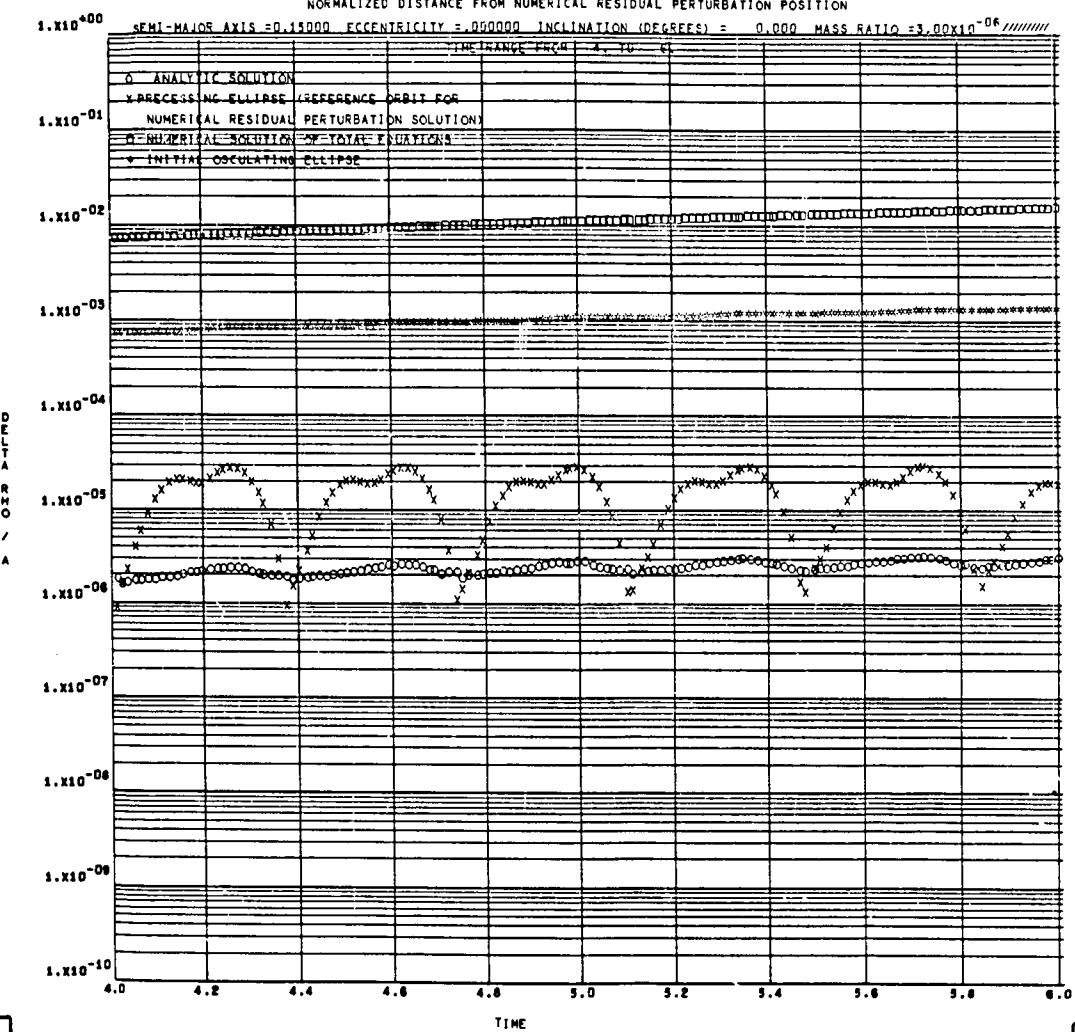


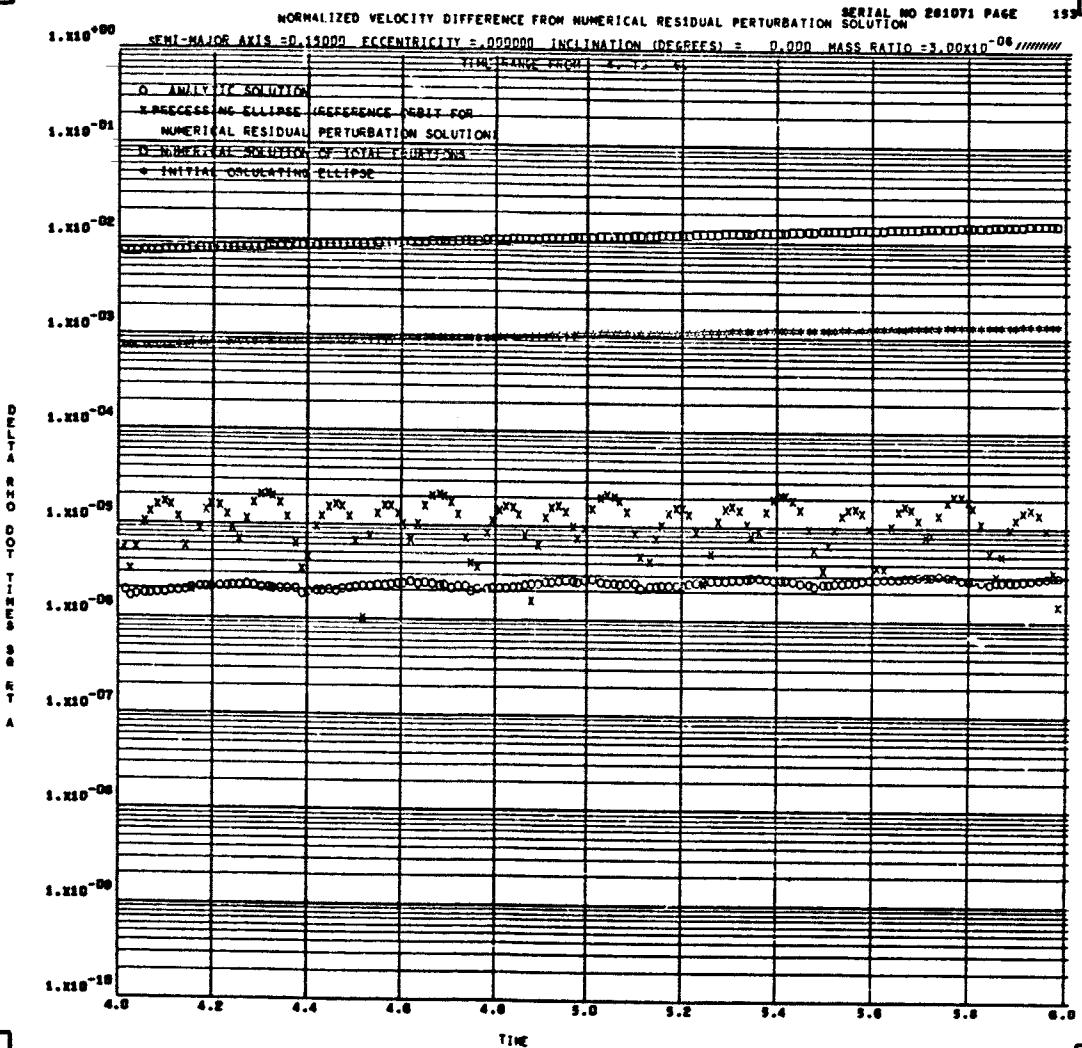
SERIAL NO 261871 PAGE 151



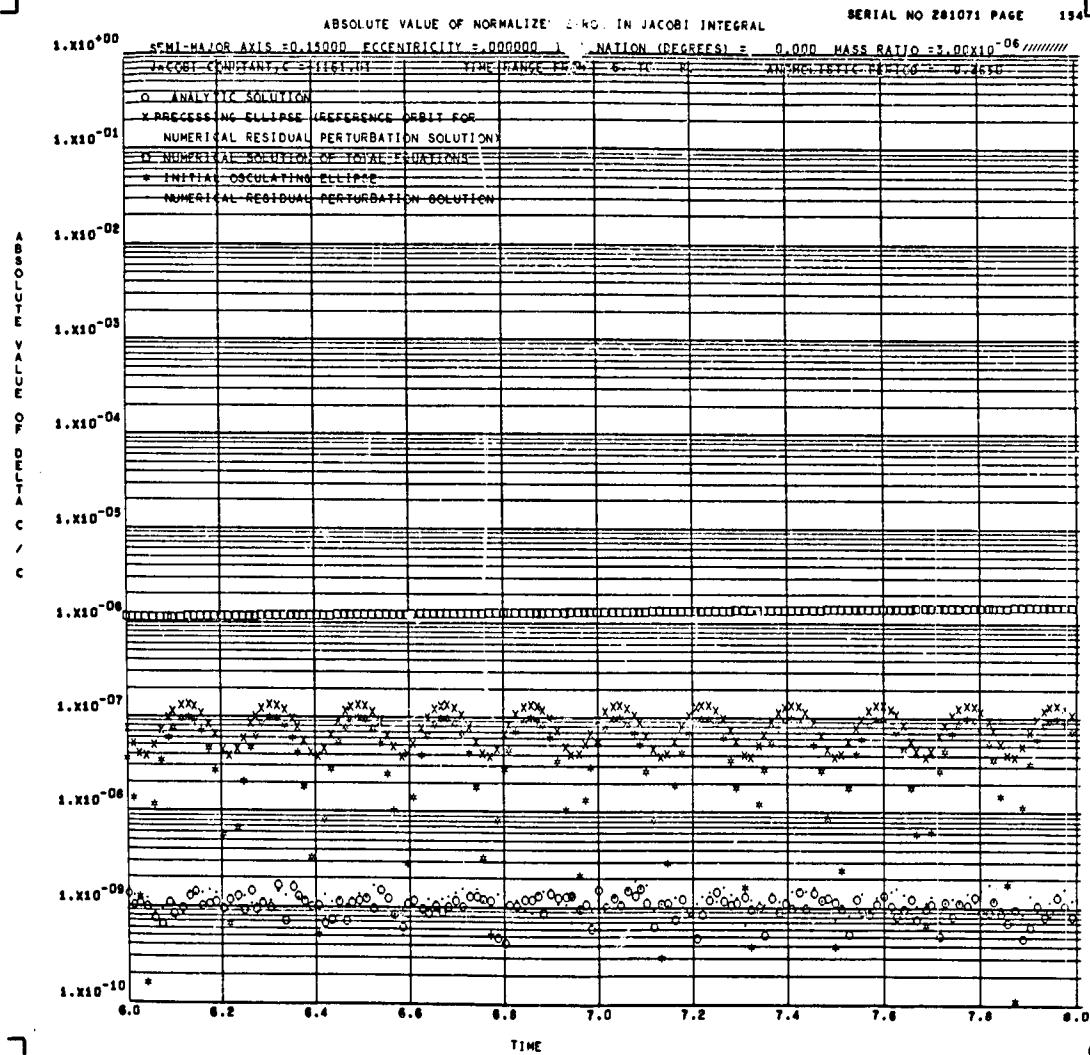
NORMALIZED DISTANCE FROM NUMERICAL RESIDUAL PERTURBATION POSITION

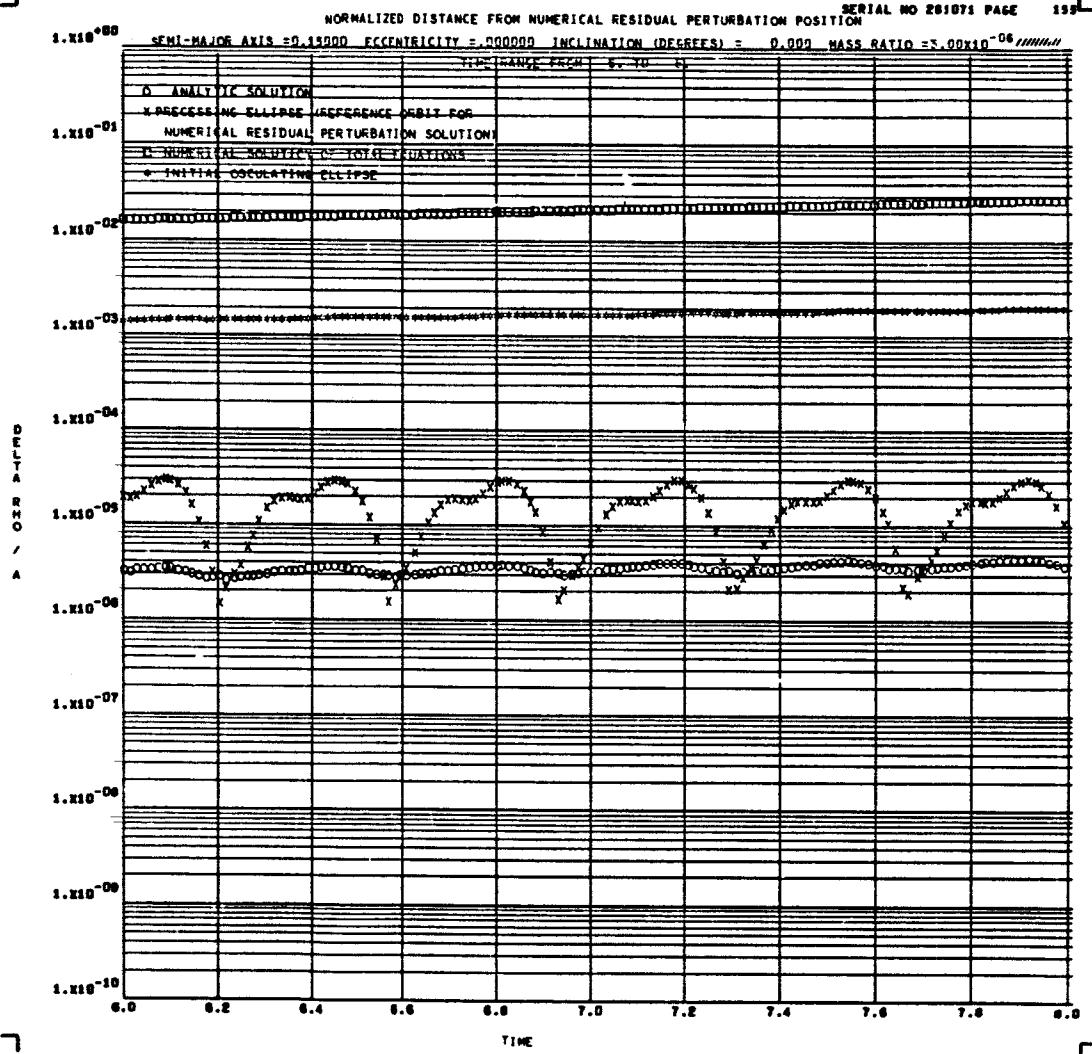
SERIAL NO 281071 PAGE 152

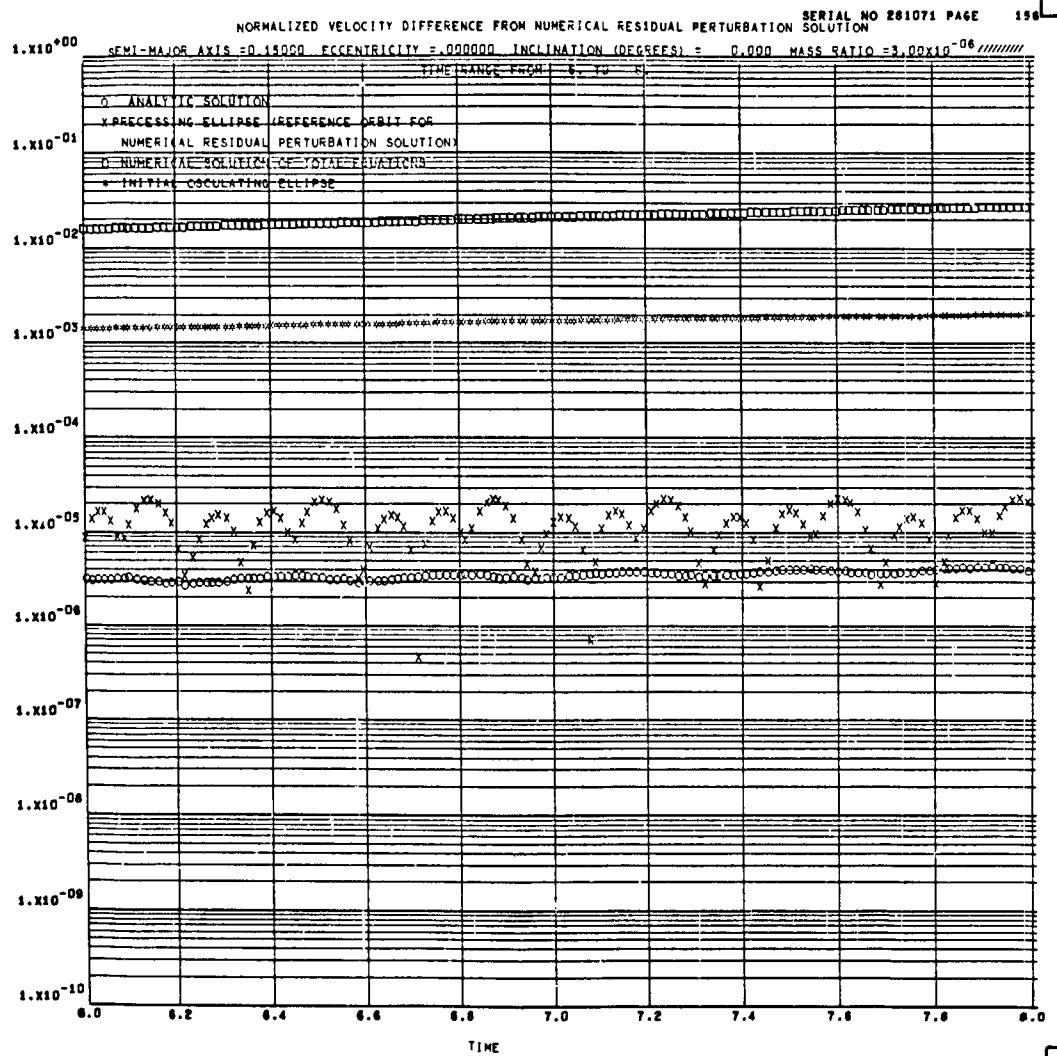


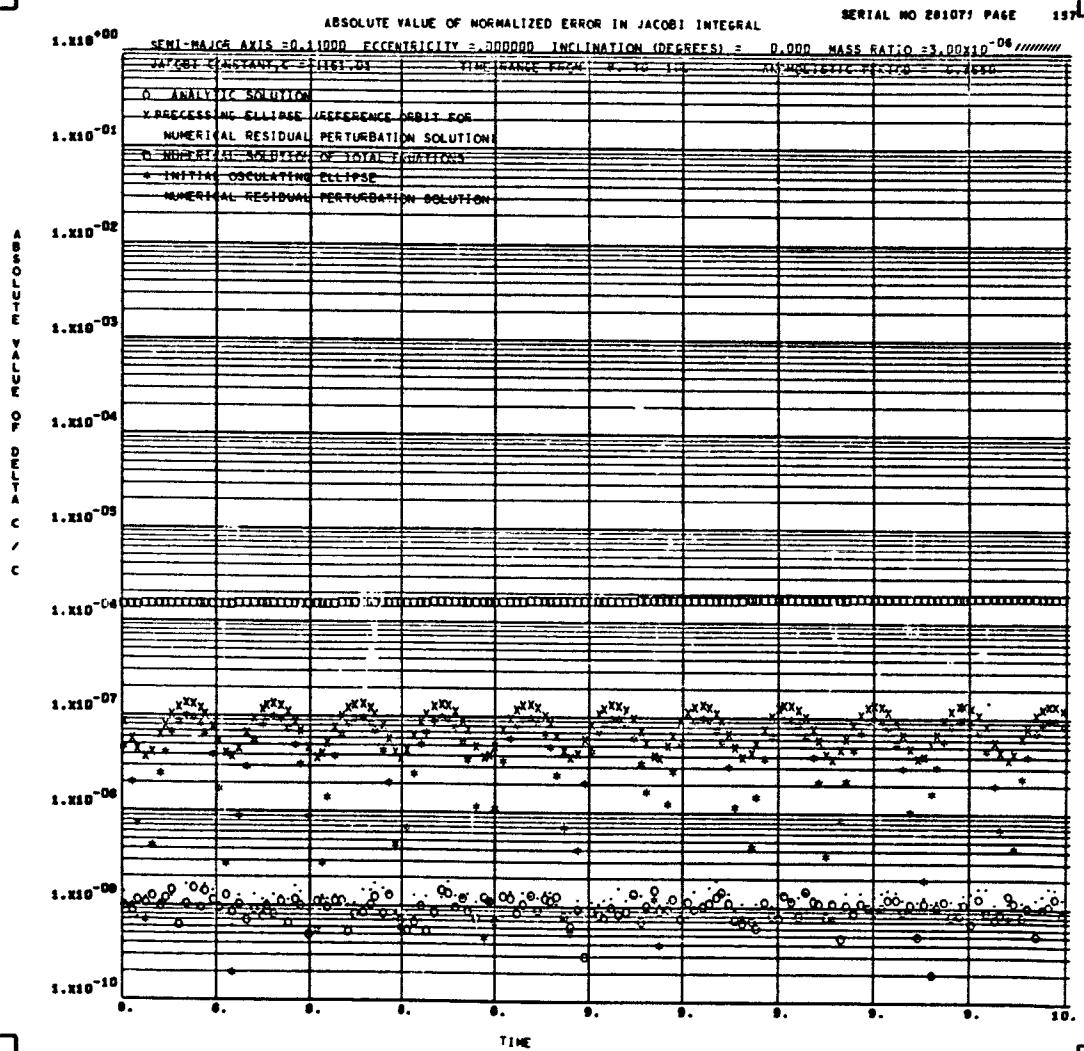


SERIAL NO 281071 PAGE 134

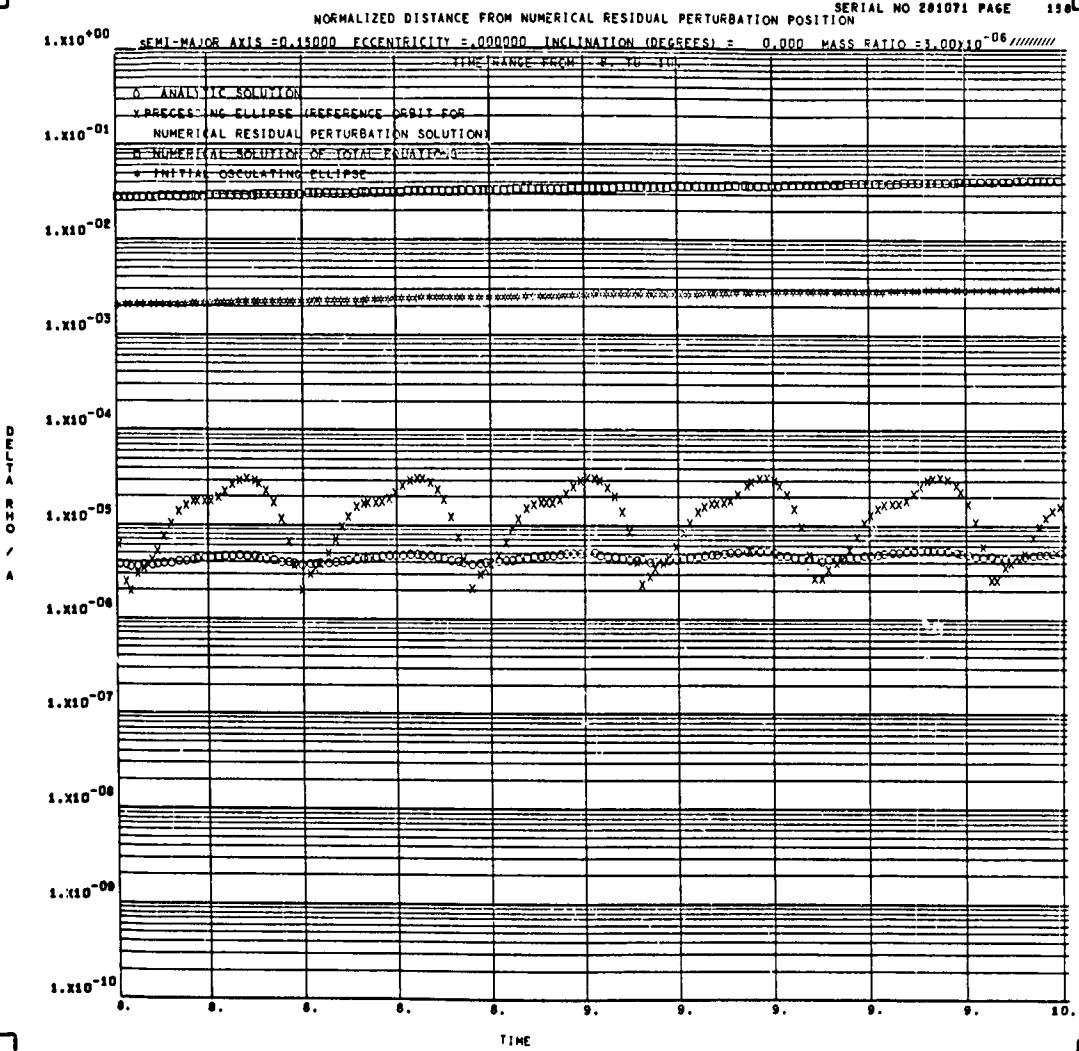






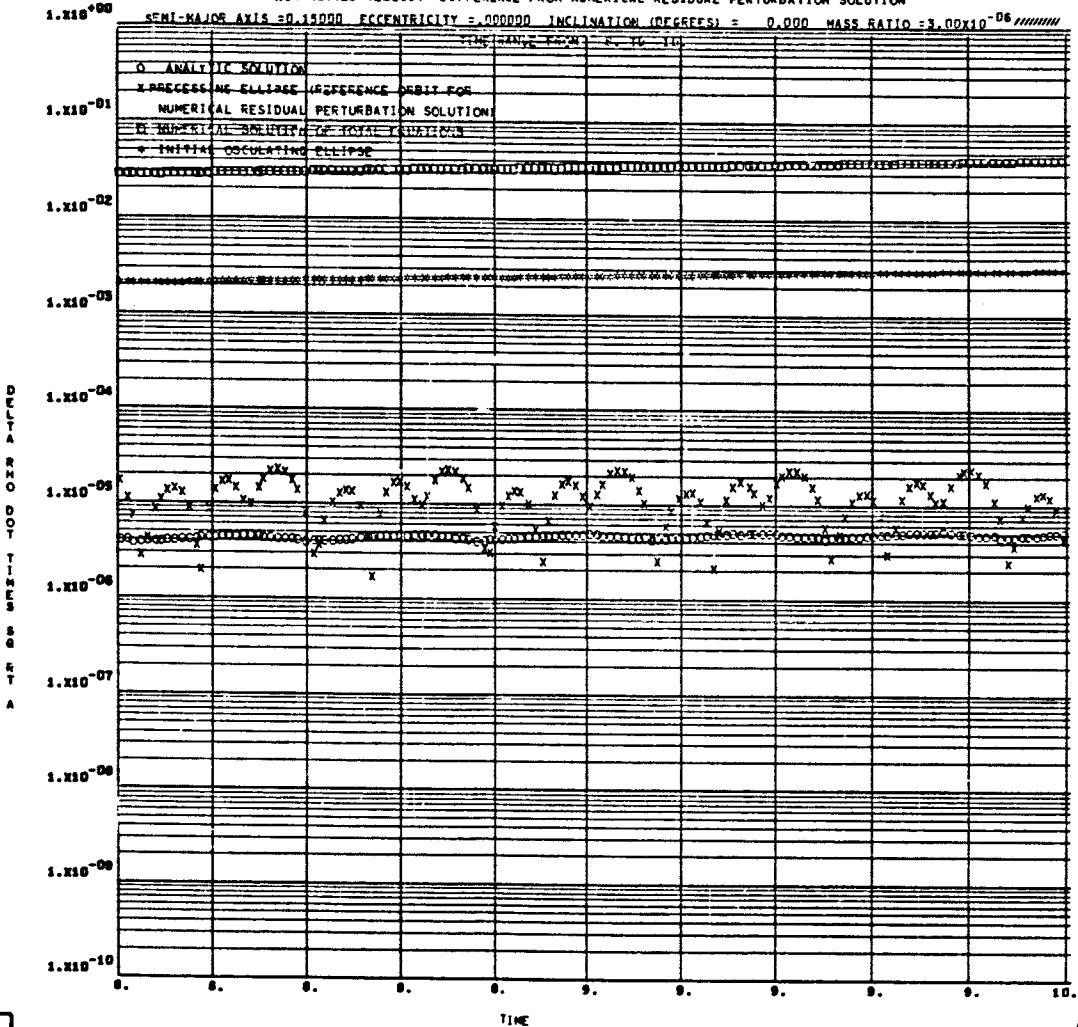


SERIAL NO 281071 PAGE 150



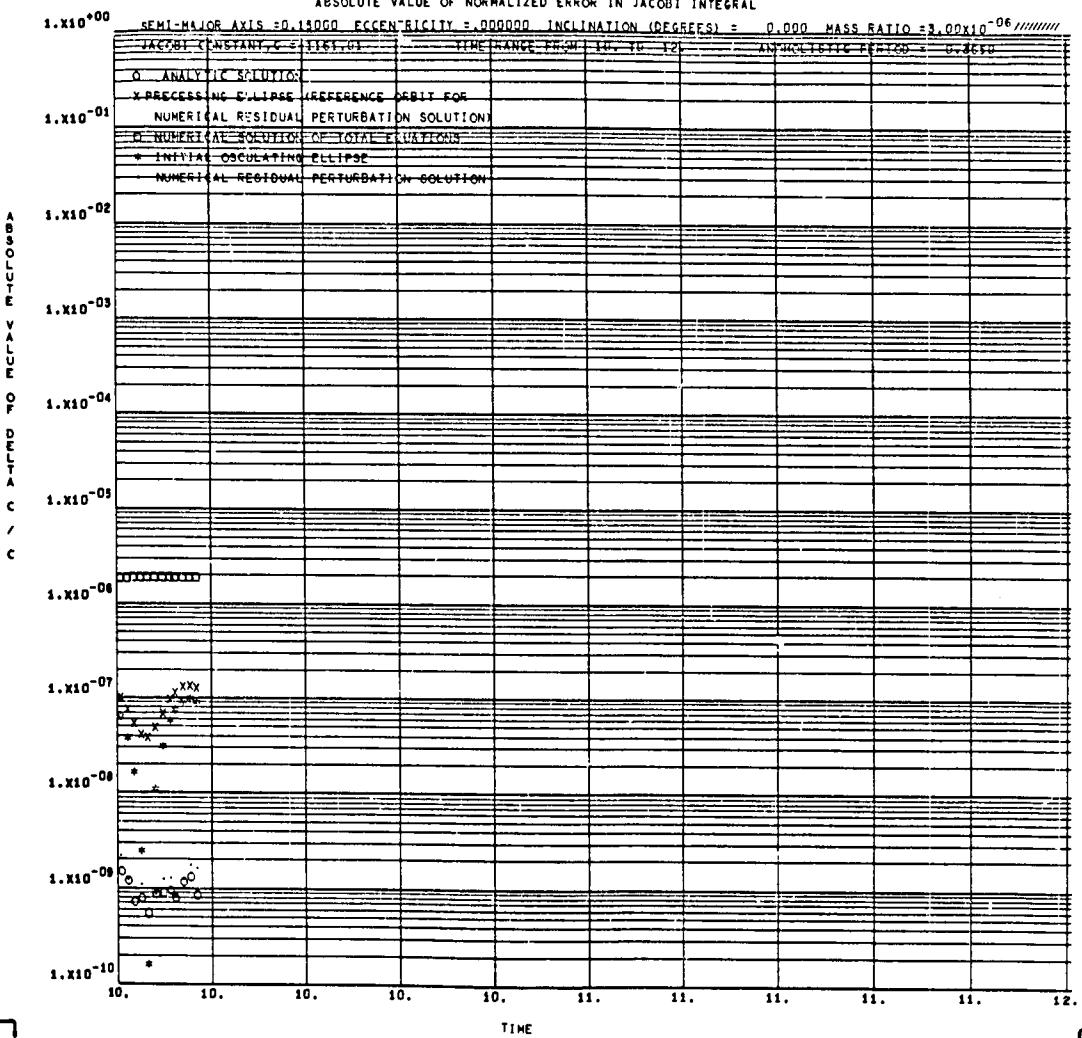
NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION SERIAL NO 281871 PAGE

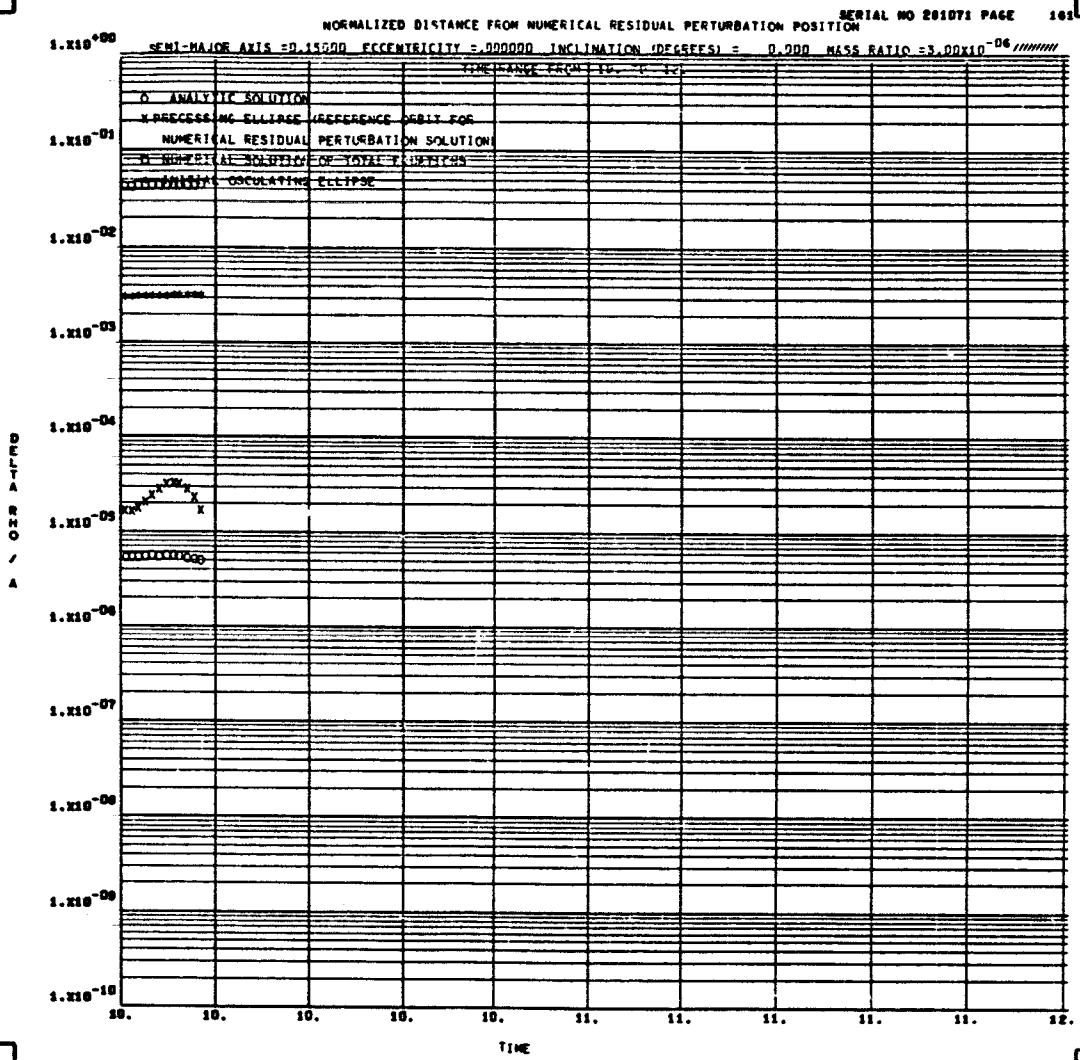
158



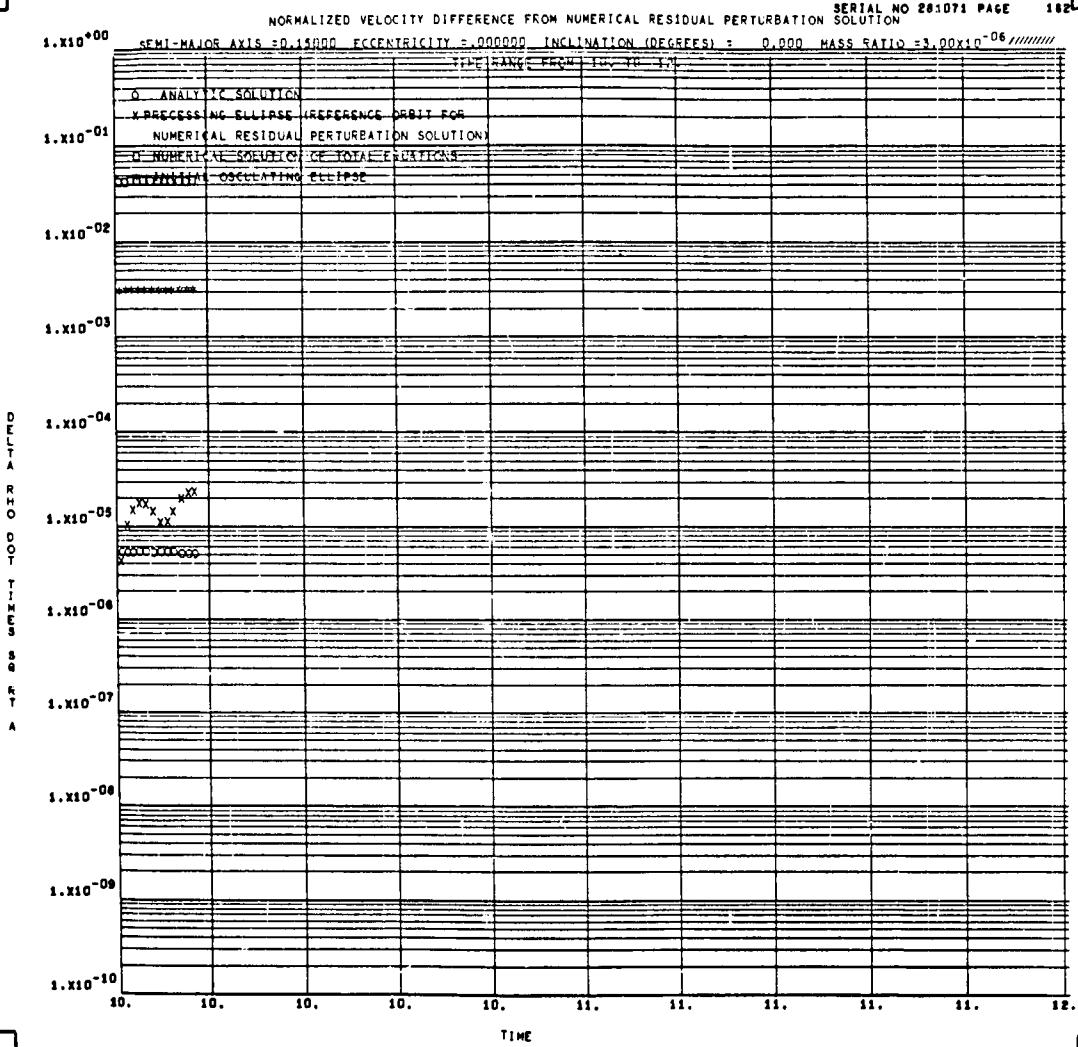
## ABSOLUTE VALUE OF NORMALIZED ERROR IN JACOBI INTEGRAL

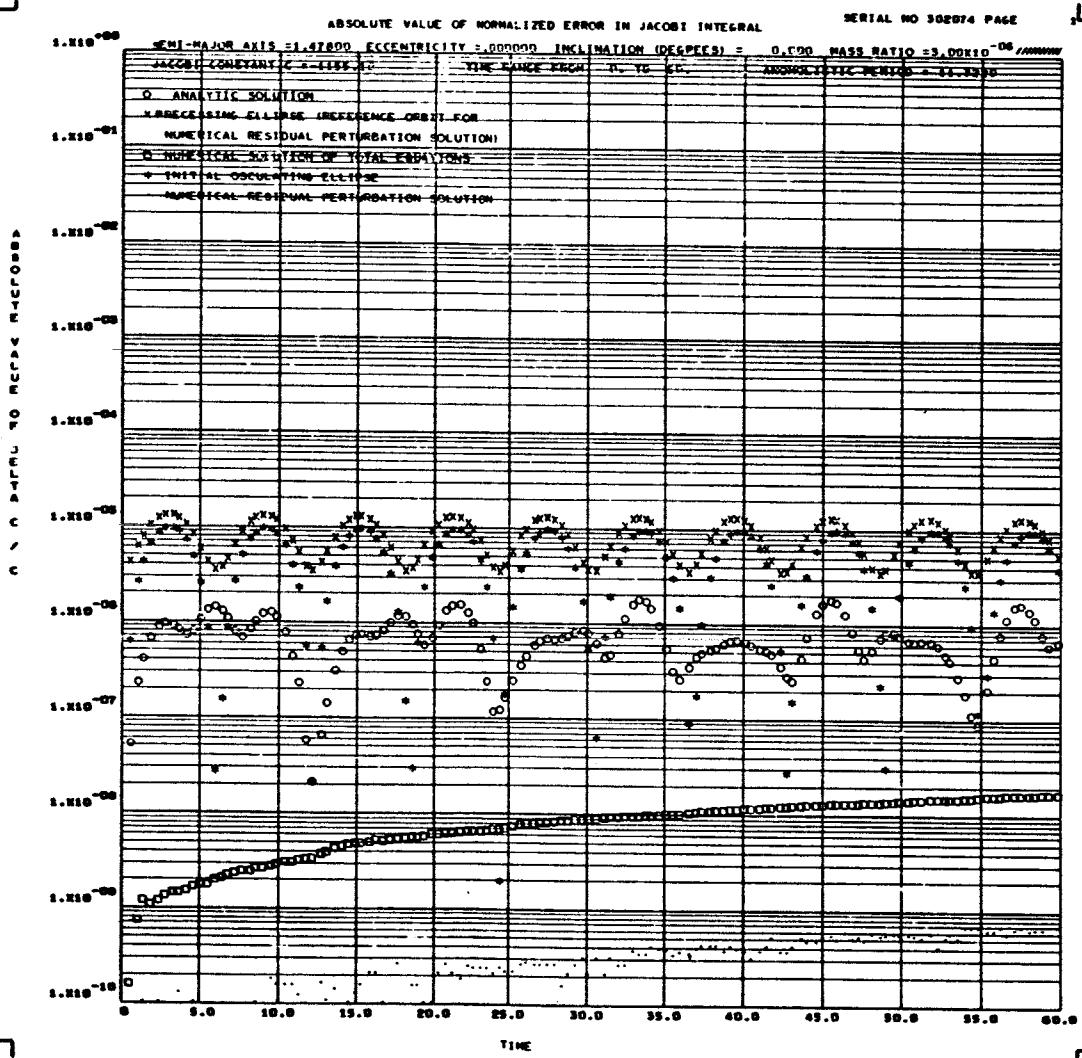
SERIAL NO 281071 PAGE 160

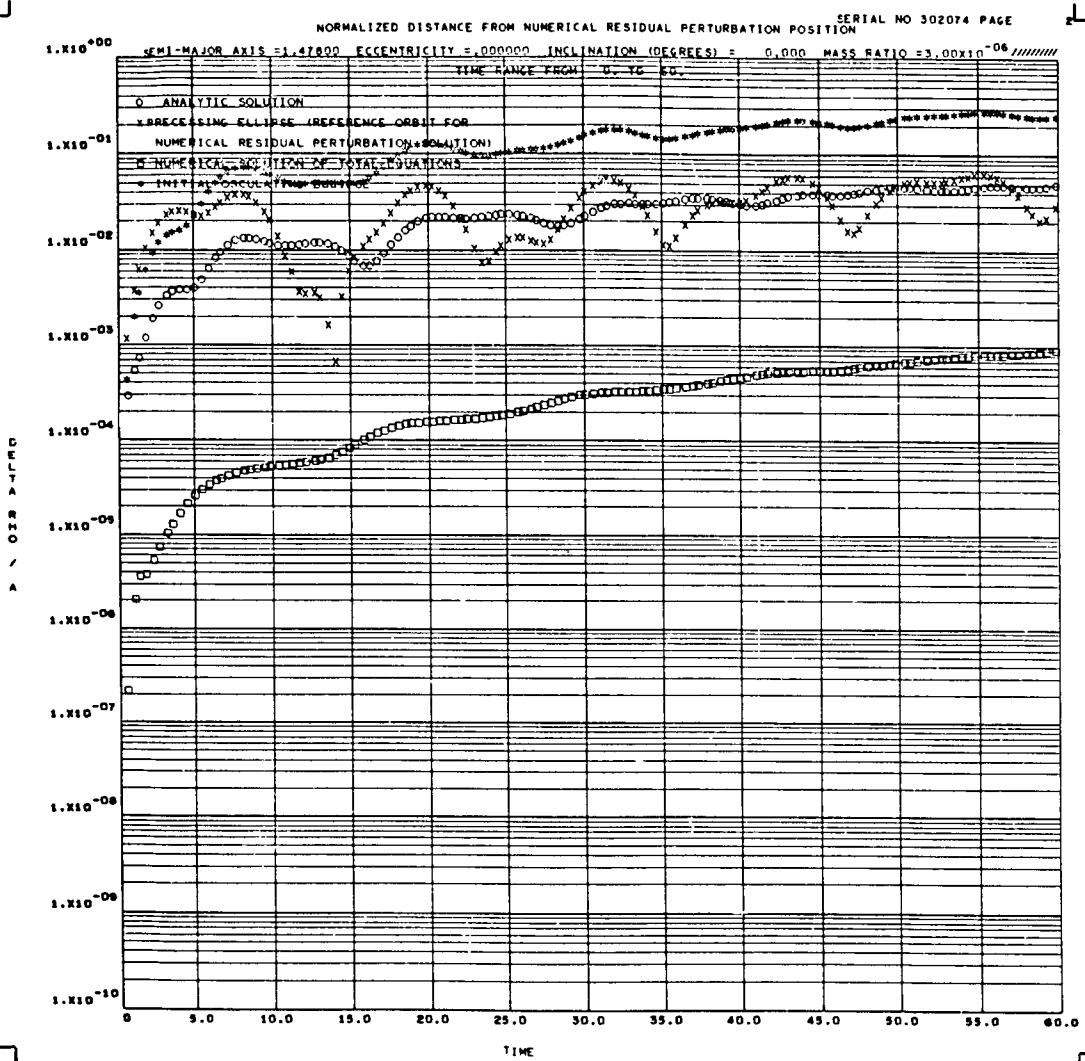


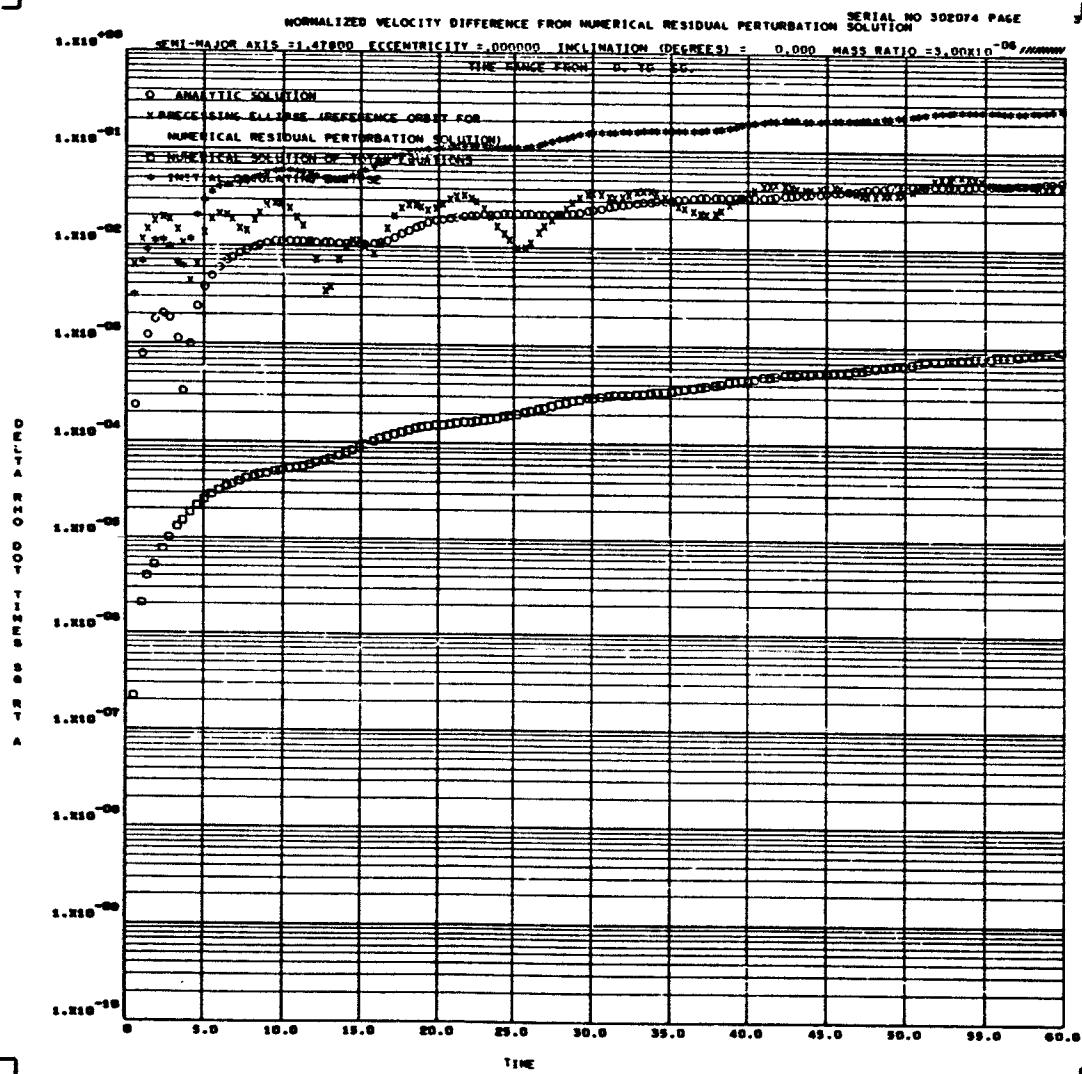


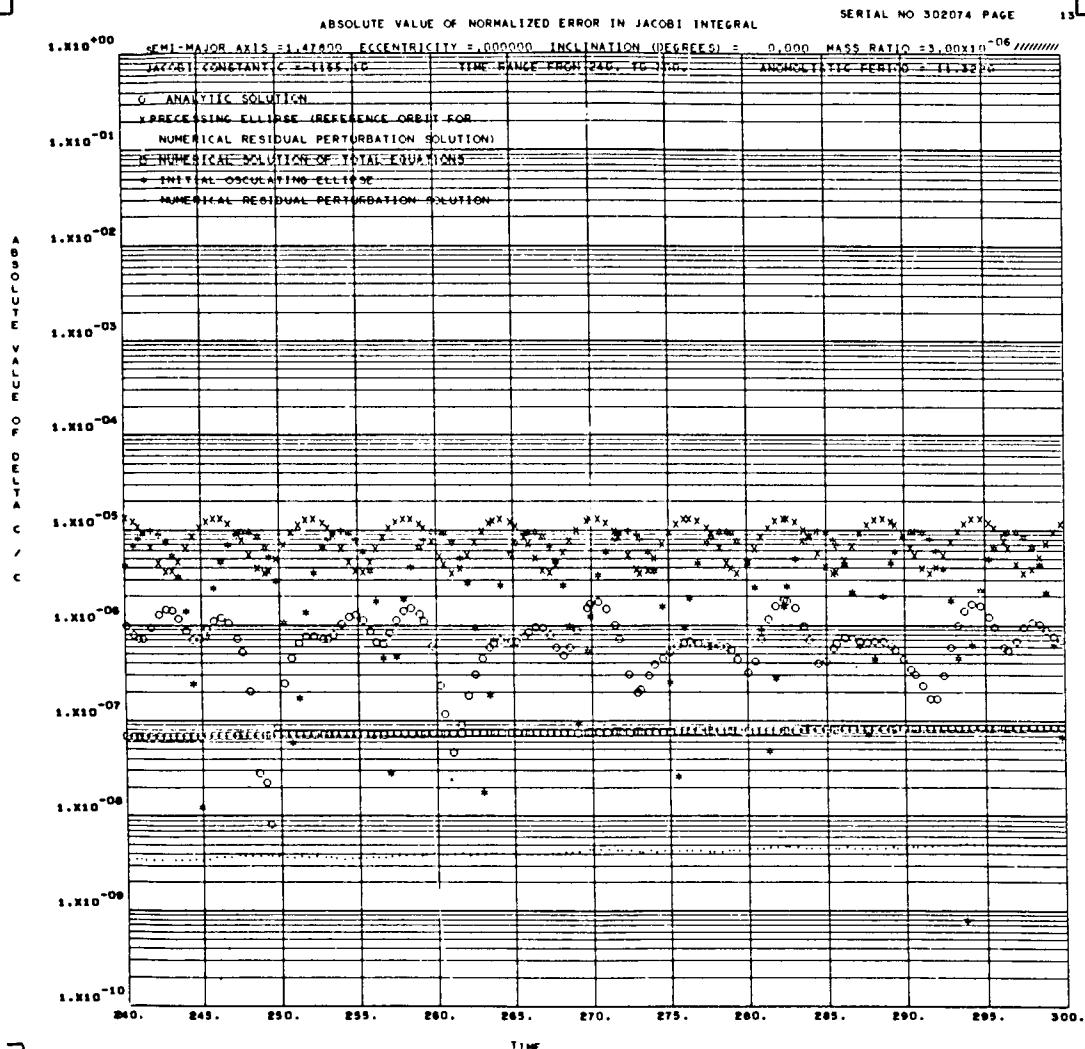
SERIAL NO 281071 PAGE 182



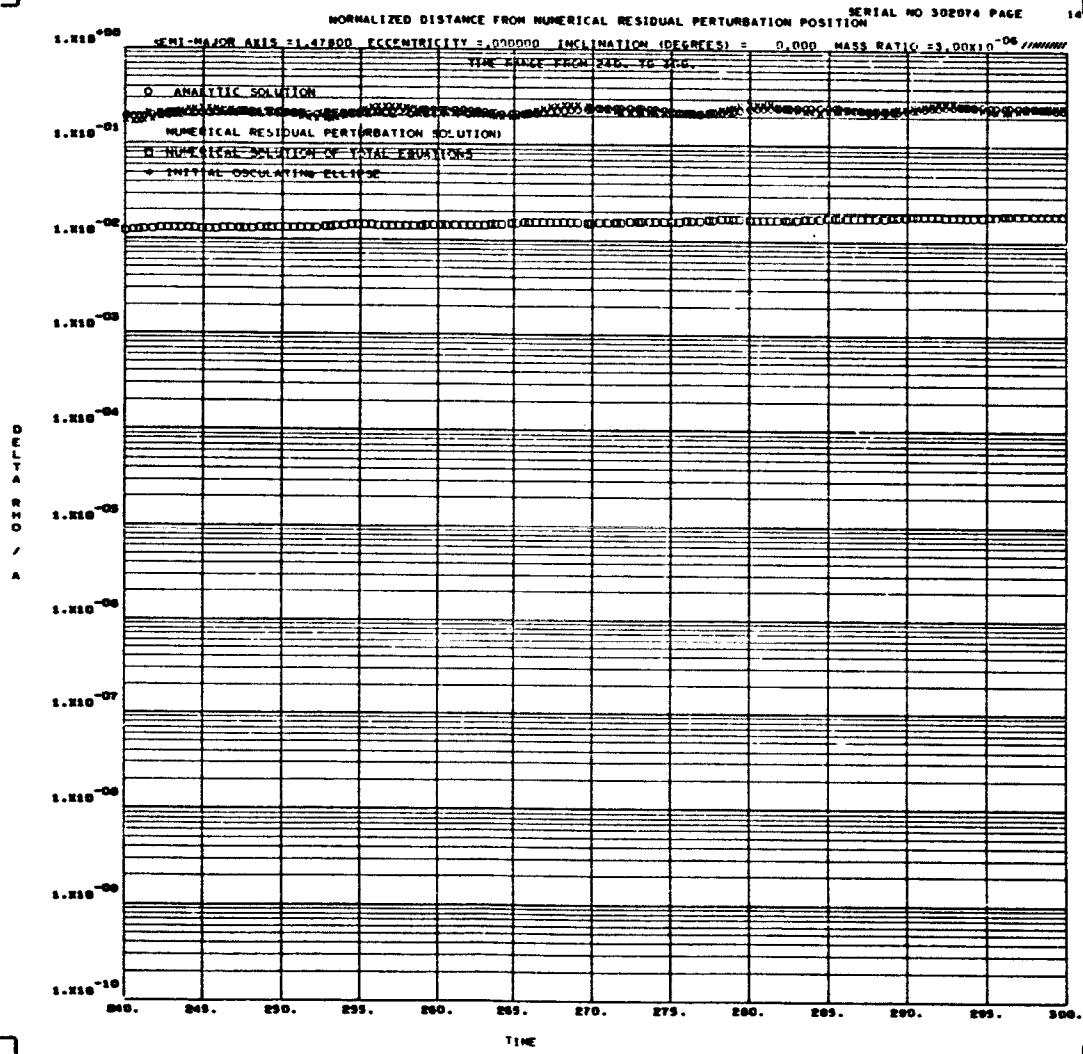




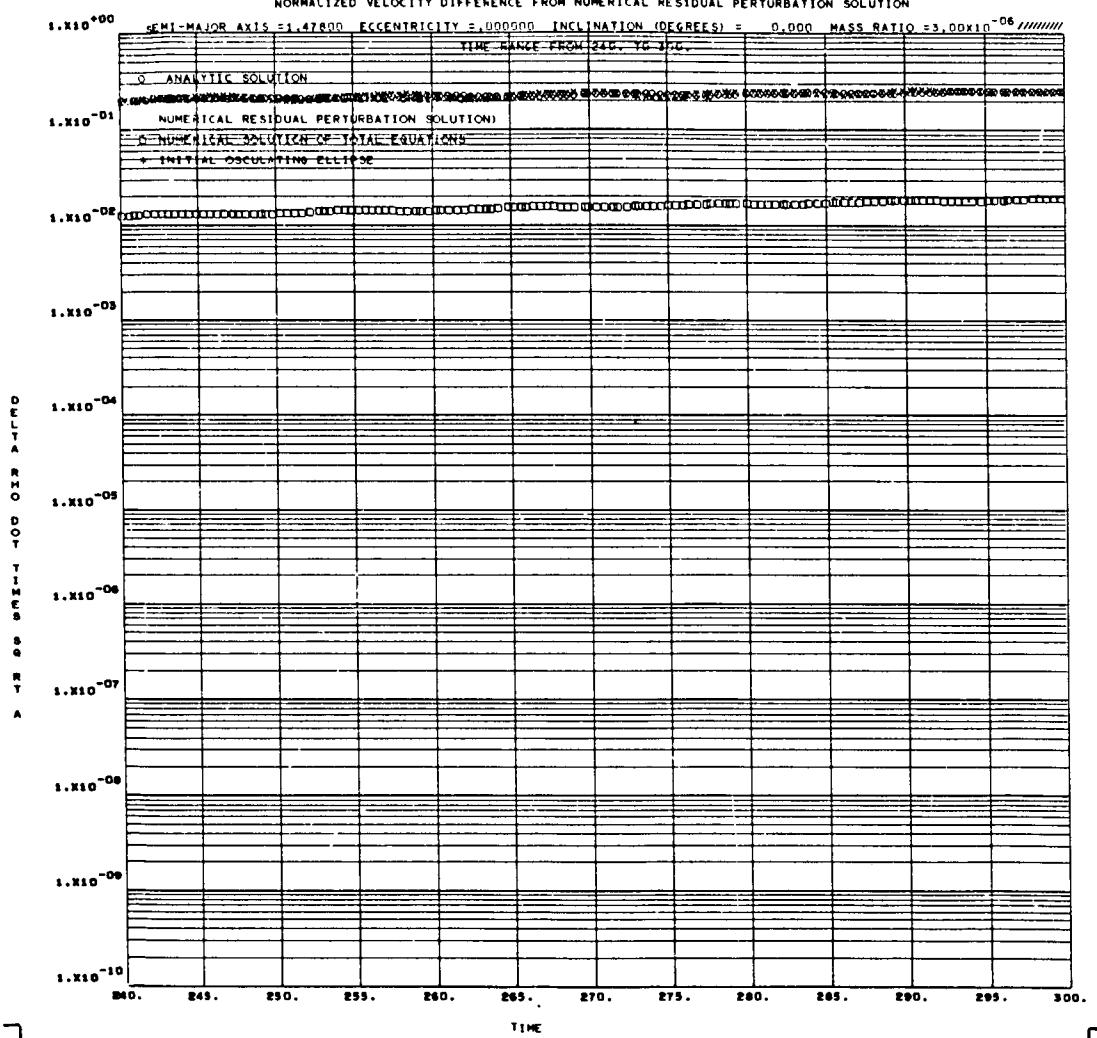


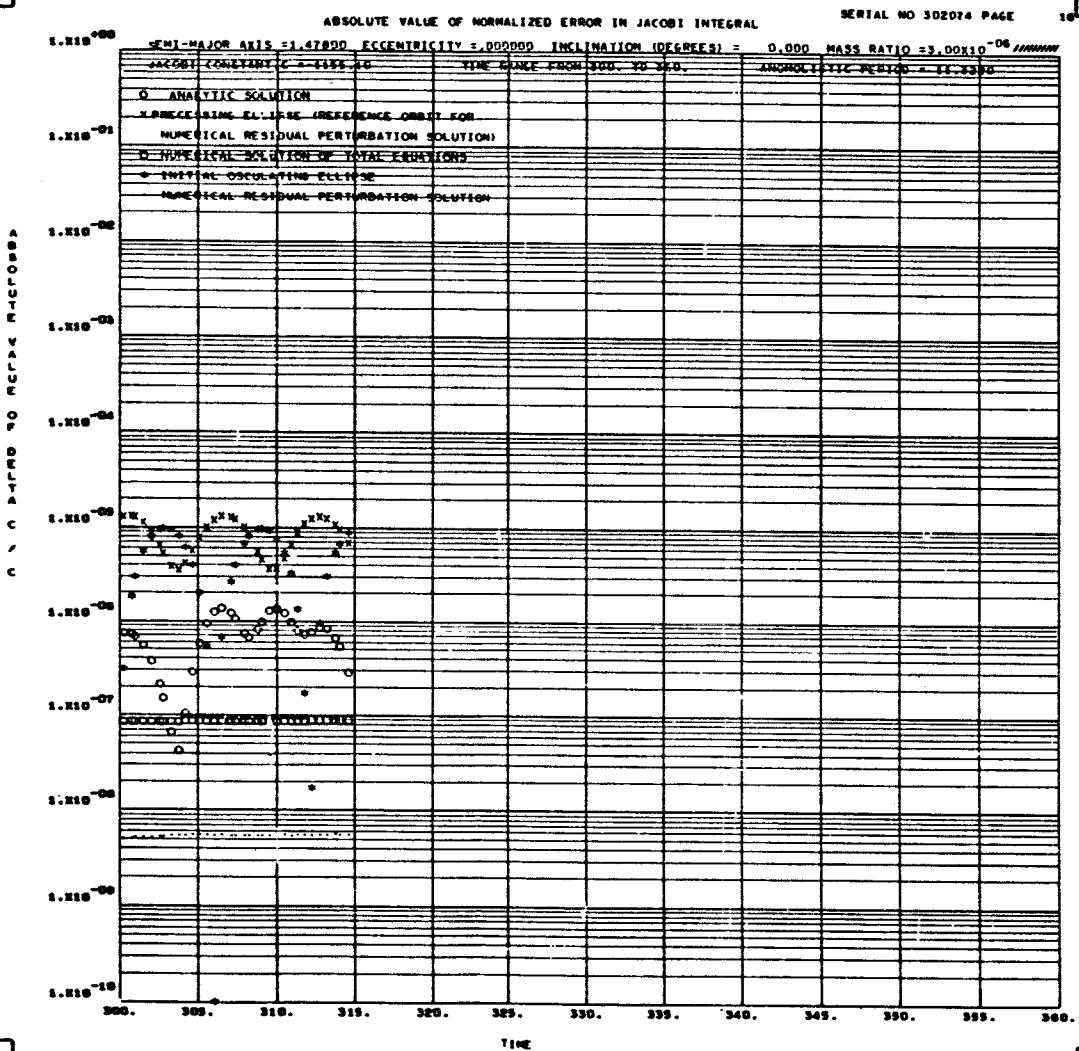


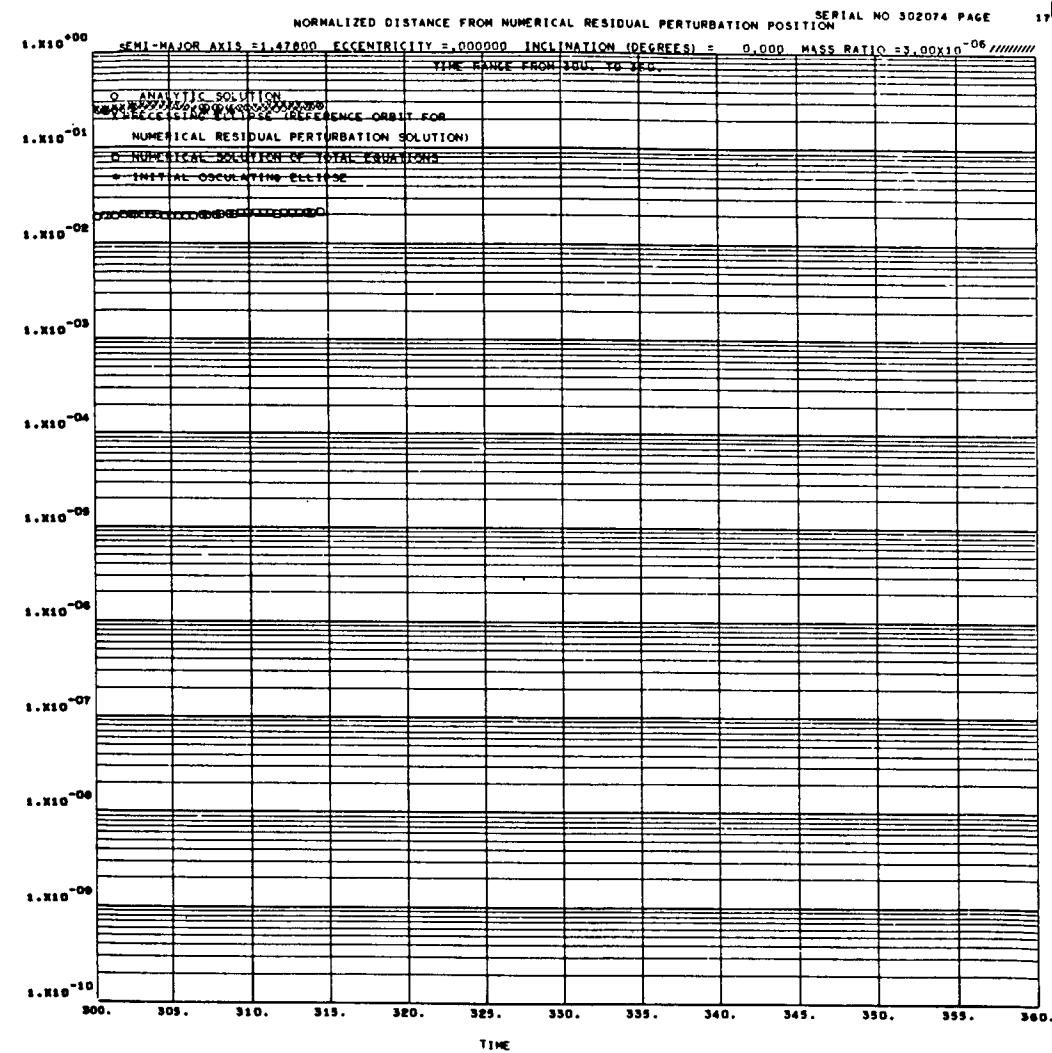
SERIAL NO 302074 PAGE 14



NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION SERIAL NO 302074 PAGE 15







NORMALIZED VELOCITY DIFFERENCE FROM NUMERICAL RESIDUAL PERTURBATION SOLUTION SERIAL NO 302074 PAGE 1

